A deep dive into the Unruh effect

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Contents

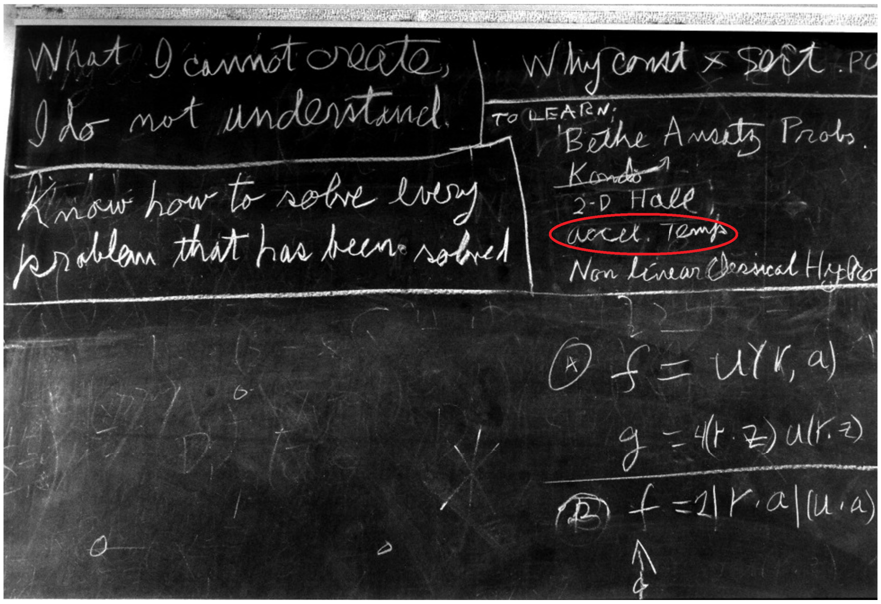

Figure 1: Richard Feynman's blackboard at the time of his death. "To learn: accel. temp."

1 Introduction

If you were to pick up your kitchen thermometer and begin accelerating in empty space with acceleration a, your thermometer would record the temperature [1, 2]

$$T = \frac{\hbar a}{2\pi c k_B} = \frac{a}{2\pi}. \tag{1.1}$$

This is the Unruh effect. (We set $\hbar = c = k_B = 1$ for the remainder of the note.) The Unruh effect relies on the fact that an accelerating observer has a different notion of "time," and therefore "frequency," and therefore "energy," and therefore "particle number," than an inertial observer. While a quantum field might be in its zero particle ground state according to an inertial observer, the same state will correspond to a thermal bath of particles according to an accelerating observer.

The goal of this document is to explain the Unruh effect from first principles. We start by defining a set of modes called "Rindler modes" which have a constant frequency according to accelerating observers. We explain how using the 'Klein-Gordon inner product,' one can construct creation and annihilation operators using these Rindler modes. These are the natural particle creation and annihilation operators which would be used by an accelerating observer to give meaning to the notion of "particle." We then note that the Rindler modes possess a curious property in relation to the usual plane wave modes with which an inertial observer would construct their creation and annihilation operators. Namely, the definition of positive and negative frequency "mix" between the Rindler modes and the plane wave modes. That is, if one creates a particular linear combination of a positive boost frequency Rindler mode on the right wedge of spacetime and a negative boost frequency Rindler mode on the left wedge of spacetime, one can make a mode which is overall positive frequency with respect to the standard inertial time coordinate. The upshot is that a certain linear combination of a Rindler annihilation operator in the right half and a Rindler creation operator in the left half can be used to make an inertial annihilation operator. This implies that the Minkowski vacuum state can be written an entangled sum of Rindler mode pairs on the right and left halves of a timeslice, respectively.

We also discuss in great detail how Rindler particles are detected by accelerating observers, and what such a detection event would look like to an inertial observer. The answer, as detailed in [3], is quite unintuitive: when an accelerating observer sees a particle *absorbed* by their detector, an inertial observer will see a particle *emitted* from the accelerating detector. We analyze this scenario in great detail using the simplified toy model of the Unruh DeWitt detector in sections 9 through 11, and also discuss a more realistic model of a "particle in a box" detector in section 13. We also discuss the subtle issue of what inertial and accelerating observers will detect at different distances from a black hole in section 15, which is often overlooked in most treatments of black hole radiation.

One section in this document of particular note is section 14, where I provide a novel account of the Unruh effect using certain properties of the ground state wave functional $\Psi_0[\phi]$. Many detailed diagrams and figures are provided in order to best illustrate how said properties of the wave functional give rise to the phenomenon of thermal entanglement between right and left.

Everything in this note is derived from first principles, and I have included extensive appendices where some of the longer calculations are carried out. For maximum simplicity, in these notes we will restrict our study to a massless real scalar field in 1+1 dimensions. My overall hope for this note is to convince the reader that the existence of the Unruh effect is not only plausible, but ultimately inevitable, and that its existence is required by the fundamental laws of quantum field theory.

2 Coordinate systems

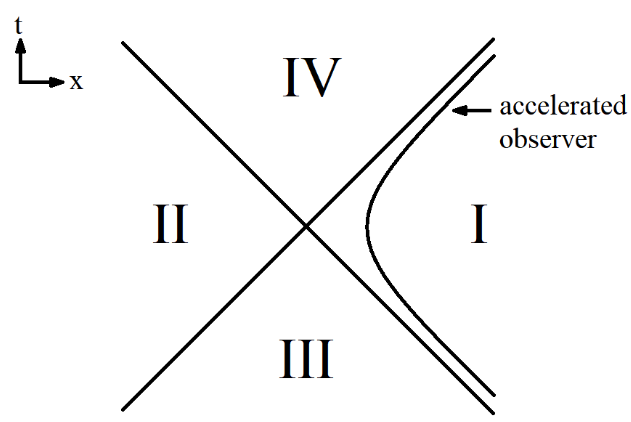

Figure 2: Four regions of Minkowski space and an accelerated Rindler observer.

Let us define three common sets of coordinates. In the standard (t, x) coordinates, the Minkowski metric is

$$ds^2 = dt^2 - dx^2. \tag{2.1}$$

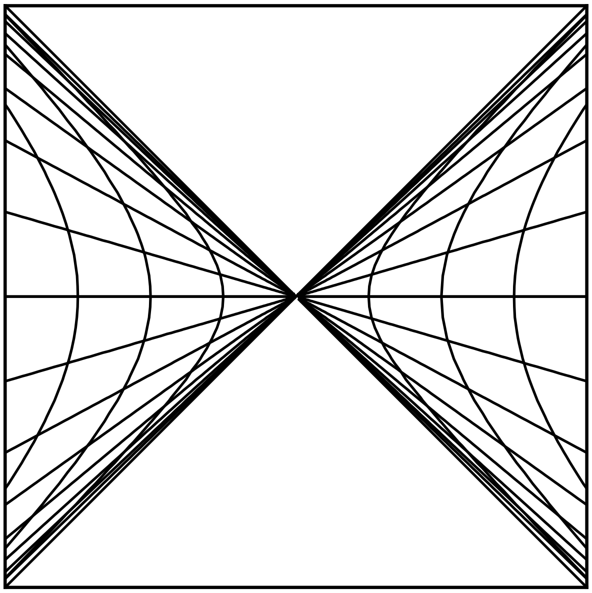

Figure 3: Rindler coordinates. The hyperbolas are lines of constant ξ and the straight lines are lines of constant η. Only regions I and II are covered.

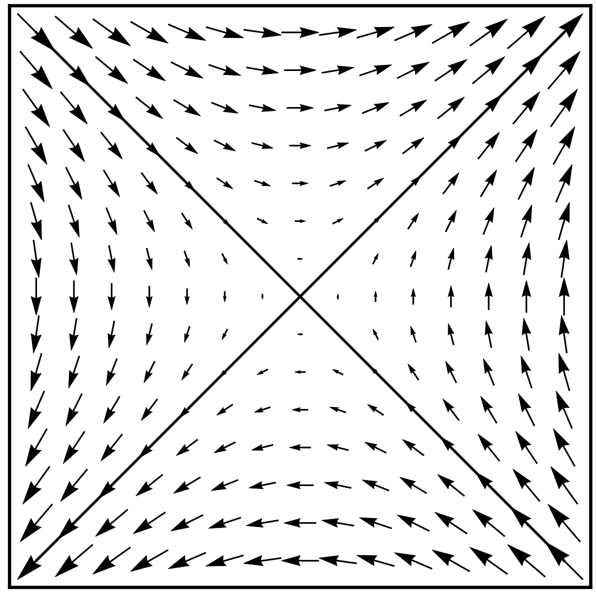

Figure 4: The boost vector field K. Note that it points backwards in region II.

In the lightcone coordinates (u, v)

$$u = t - x \tag{2.2}$$

$$v = t + x \tag{2.3}$$

the metric is

$$ds^2 = du dv. \tag{2.4}$$

In Rindler coordinates (η, ξ)

$$t = \xi \sinh(\eta) \tag{2.5}$$

$$x = \xi \cosh(\eta) \tag{2.6}$$

the metric is

$$ds^2 = \xi^2 d\eta^2 - d\xi^2. \tag{2.7}$$

In Rindler coordinates, an observer traveling with a constant proper acceleration a will trace out a path $(\eta, \xi) = (a\tau, 1/a)$ where τ is their proper time. We call η the "boost time" or "Rindler time." ξ corresponds to the proper distance to the origin. If $\xi > 0$, these coordinates cover the right Rindler wedge, i.e. Region I. If $\xi < 0$, they cover the left Rindler wedge, i.e. Region II. Regions III and IV are not covered by Rindler coordinates, as shown in Figure 3.

The "boost" vector field K is a Killing symmetry of our spacetime. In all three coordinate systems it is given by

$$\begin{aligned} K = K^\mu \partial_\mu &= t\partial_x + x\partial_t \\ &= -u\partial_u + v\partial_v \\ &= \partial_\eta . \end{aligned} \tag{2.8}$$

As shown in Figure 4, the boost vector field points towards the future in Region I, but points towards the past in Region II. (In section 8 we will use this fact to explain the phenomenon of the mixing of positive and negative frequency modes.)

3 Rindler modes

In this section we will define and study Rindler modes as solutions to the classical wave equation. In later sections we will quantize the theory.

Say we have a massless real scalar field in 1+1 dimensions with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\nabla^\mu \phi \nabla_\mu \phi. \tag{3.1}$$

The equation of motion is

$$0 = \nabla^\mu \nabla_\mu \phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu \phi). \tag{3.2}$$

In the (t, x) coordinates, the equation of motion is

$$\begin{aligned} 0 &= (\partial_t^2 - \partial_x^2)\phi \\ &= (\partial_t + \partial_x)(\partial_t - \partial_x)\phi. \end{aligned} \tag{3.3}$$

Any solution to this equation can be written as the sum of an arbitrary function of $(t - x)$ and an arbitrary function of $(t + x)$. Therefore, every solution can be written as the sum of a "right moving" part and a "left moving" part.

$$\phi(t, x) = f_R(t - x) + f_L(t + x) \tag{3.4}$$

In the lightcone coordinates, the equation of motion is

$$0 = \partial_u \partial_v \phi \tag{3.5}$$

and once again solutions are of the form

$$\phi(u, v) = f_R(u) + f_L(v). \tag{3.6}$$

In Rindler coordinates, the equation of motion is

$$\begin{aligned} 0 &= \xi^{-2}(\partial_\eta^2 \phi - \xi^2 \partial_\xi^2 - \xi\partial_\xi)\phi \\ &= \xi^{-2}(\partial_\eta^2 \phi - (\xi\partial_\xi)^2)\phi \\ &= \xi^{-2}(\partial_\eta + \xi\partial_\xi)(\partial_\eta - \xi\partial_\xi)\phi \end{aligned} \tag{3.7}$$

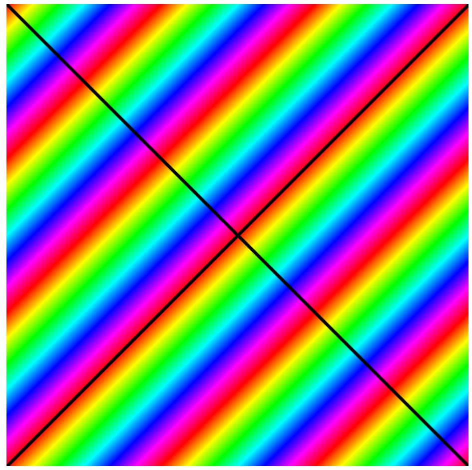

Figure 5: This is the solution $e^{-i(|k|t-kx)}$. We have chosen $k > 0$, so the plane wave is "right moving." If we were to have drawn the complex conjugate mode $e^{i(|k|t-kx)}$ instead, then the order in which the colors are cycled through would reverse. For instance, tracing a line up the page, we can see the colors go from red to blue to green. For the complex conjugate mode, they would go from green to blue to red.

Solutions are therefore of the form

$$\phi(\eta, \xi) = f_R(\eta - \log \xi) + f_L(\eta + \log \xi). \tag{3.8}$$

It is often useful to write our solutions in terms of linear combinations of plane waves with complex coefficients. For instance, we may write

$$\phi(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dk}{2|k|} (a_k e^{-i(|k|t-kx)} + a_k^* e^{i(|k|t-kx)}) \tag{3.9}$$

where a_k and a_k^* are complex coefficients which determine $\phi(t, x)$. Note that $e^{-i(|k|t+kx)}$ is called a "positive frequency mode" and $e^{i(|k|t-kx)}$ is called a "negative frequency mode." (In general parlance, a "mode" is just a function that solves the wave equation such that an arbitrary field ϕ can be written over as a sum over such modes.)

Note that we can write our right moving and left moving functions as sums over the positive and negative "momenta" k respectively

$$f_R(t - x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dk}{2|k|} \Theta(k)(a_k e^{-i(|k|t-kx)} + a_k^* e^{i(|k|t-kx)}) \tag{3.10}$$

$$f_L(t + x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dk}{2|k|} \Theta(-k)(a_k e^{-i(|k|t-kx)} + a_k^* e^{i(|k|t-kx)}). \tag{3.11}$$

where Θ is the Heaviside step function.

One may wonder how we can picture these plane wave modes $e^{-i(|k|t-kx)}$. One way is to use color to represent the phase of the wave, allowing us to represent the value of the wave at every point in spacetime. This is what we have done in Figure 5.

Consider an inertial observer stationed on the worldline $x = 0$. On their worldline, they will see the value of this mode oscillate as $e^{-i|k|t}$, i.e. with a constant frequency $|k|$. When we

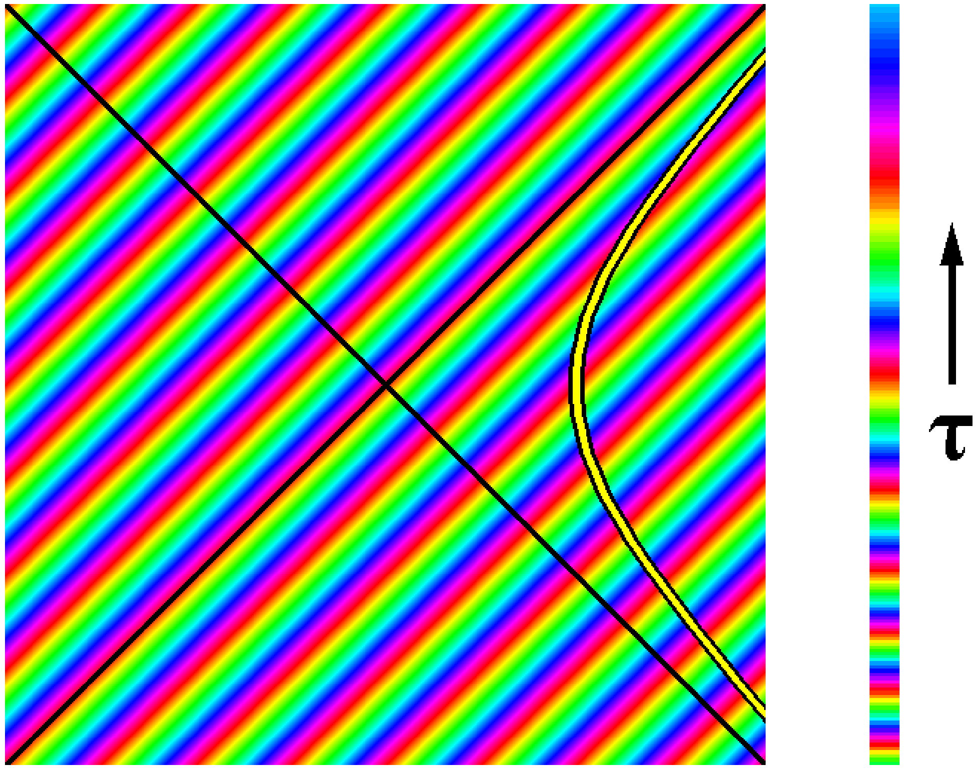

Figure 6: Here, we depict the plane wave $e^{-i(|k|t-kx)}$ and show an accelerating observer moving along a hyperbola. This observer sees the phase of the plane wave oscillate slower and slower as their proper time τ increases, meaning that the plane wave does not have a constant frequency to the accelerated observer.

quantize our field, each mode will have a particle occupation number associated with it, and the energy of a particle in the mode with frequency $|k|$ is $\hbar|k|$. Therefore, we should feel welcome to adopt the terminology that $|k|$ is the "energy" of the mode. We see that constant frequencies in time are the same thing as definite energies. If the observer at $x = 0$ can build a device which can detect particles with an energy of $|k|$, such a device must couple to the $e^{-i|k|t}$ mode somehow.

Now, how would an accelerating observer see one of these plane waves oscillate on their own worldline? This has been depicted in is depicted in Figure 6. They do not see the phase increase with a constant frequency as the inertial observer did. As their proper time τ increases, the frequency of oscillation decreases. Early in the journey, the wave oscillates very quickly both because the wave front of the plane wave is passing the observer perpendicularly to their wordline, and because their proper time τ is progressing slowly due to time dilation (according to the stationary observer). Late in the accelerating observer's journey, however, they will see the wave oscillate very slowly because the path becomes essentially parallel to the wave's profile. So, early in the journey, the frequency is arbitrarily high and late in the journey the frequency is arbitrarily low.

So, which modes have constant frequency according to the accelerated observer's time coordinate η? These modes can be found using Eq. 3.8. The right moving positive frequency mode

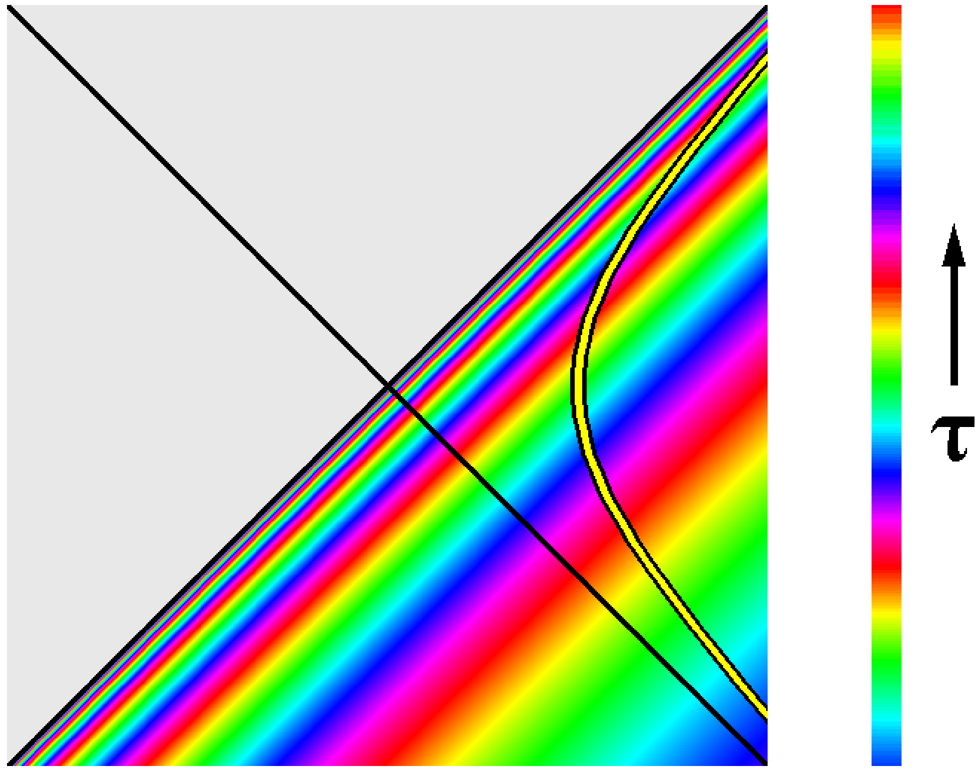

Figure 7: Here we depict the right moving Rindler mode $e^{-i\lambda(\eta - \log \xi)} = (-u)^{i\lambda}$ and show an accelerating observer moving along a hyperbola. The accelerated observer sees the Rindler mode as having a constant frequency with respect to their proper time τ. (The Rindler mode is 0 in regions II and IV.)

(which has support in the right Rindler wedge where $u < 0$) is given by

$$e^{-i\lambda(\eta - \log \xi)} = e^{i\lambda \log(-u)} = (-u)^{i\lambda} \qquad \text{(region I)} \tag{3.12}$$

where λ (with $\lambda > 0$) is the "Rindler frequency." In Figure 7 we draw a modified version of 6 where the accelerated observer sees a Rindler mode instead of a plane wave. Note that the rate of oscillation becomes infinite on the acceleration horizon as $u \to 0$. This is necessary for the accelerated observer to see the frequency remain constant.

To summarize: we have seen that because the inertial and accelerated observers use different notions of time, they also have different notions of frequency. (Upon quantization, this will also imply that they have different notions of what a particle is.)

4 Field operator conventions

When quantizing the field, we have the field operator $\hat{\phi}(x)$ as well as its conjugate momentum $\hat{\pi}(x)$. They satisfy the commutation relation

$$[\hat{\phi}(x), \hat{\pi}(x')] = i\delta(x - x'). \tag{4.1}$$

We define the free Hamiltonian of the quantum field, denoted $H_0^{\rm QFT}$, to be

$$H_0^{\rm QFT} = \frac{1}{2}\int dx\, \hat{\pi}(x)^2 - E_0 \tag{4.2}$$

where the constant energy E_0 is chosen such that the energy of the Minkowski ground state is 0, i.e.

$$H_0^{\rm QFT} |0\rangle = 0. \tag{4.3}$$

We now define the time evolved field operators by

$$\hat{\phi}(t,x) \equiv e^{itH_0^{\rm QFT}} \hat{\phi}(x) e^{-itH_0^{\rm QFT}} \tag{4.4}$$

$$\hat{\pi}(t,x) \equiv e^{itH_0^{\rm QFT}} \hat{\pi}(x) e^{-itH_0^{\rm QFT}} \tag{4.5}$$

which have the equal time commutation relation

$$[\hat{\phi}(t,x), \hat{\pi}(t,x')] = i\delta(x-x'). \tag{4.6}$$

5 Introduction the Klein-Gordon inner product

All introductory courses in quantum field theory define creation and annihilation operators. Such creation and annihilation operators, however, only create and annihilate modes which are *plane waves* of the form $e^{-i|k|t+ikx}$, given by the standard mode decomposition (3.9). However, we have argued that an accelerated observer's natural notion of constant frequency modes will not be plane waves, but will instead be solutions called "Rindler modes." Therefore, we need to find a more general way to define creation and annihilation operators. To do this, we will use Klein-Gordon inner product, a mathematical object which can be used to generalize the construction of creation and annihilation operators to modes other than the usual plane waves. We will review all of the useful properties of the Klein-Gordon inner product here, which shall be used repeatedly throughout the rest of the note.

The Klein-Gordon inner product is defined by

$$(f,g)_{\rm KG} = i\int dx\, \big(f^*(\partial_t g) - g(\partial_t f^*)\big). \tag{5.1}$$

It is an integral over x for $t = $ const. If f and g both solve the wave equation, $(\partial_t^2 - \partial_x^2)f = (\partial_t^2 - \partial_x^2)g = 0$, then the KG inner product is independent of time, i.e. it doesn't matter *which* value t is taken to be. This is simple to verify:

$$\partial_t (f,g)_{\rm KG} = i\int dx \big((\partial_t f^*)(\partial_t g) + f^*(\partial_t^2 g) - (\partial_t g)(\partial_t f^*) - g(\partial_t^2 f^*)\big) \tag{5.2}$$

$$= i\int dx \big(-f^*(\partial_x^2 g) + g(\partial_x^2 f^*)\big) \tag{5.3}$$

$$= 0 \tag{5.4}$$

where in the final step we integrated by parts and assumed that f and g decay sufficiently rapidly at infinity.

The main purpose of the KG inner product is to define creation and annihilation operators. If f is a function which saties the wave equation, then we define $\hat{a}^\dagger(f)$ and $\hat{a}(f)$ as

$$\hat{a}^\dagger(f) \equiv -(f^*, \hat{\phi})_{\mathrm{KG}} \tag{5.5}$$

$$\hat{a}(f) = (f, \hat{\phi})_{\mathrm{KG}}. \tag{5.6}$$

(Notice that the field operator $\hat{\phi}$ of a free field satisfies the wave equation $(\partial_t^2 - \partial_x^2)\hat{\phi} = 0$, so it can be placed in the KG inner product.)

As a brief sanity check, let's see that if $f = e^{-i|k|t+ikx}$ that we recover the standard formulae for the creation and annihilation operators. Here, we will evaluate the KG inner product on the $t = 0$ slice:

$$\begin{aligned} \hat{a}^\dagger(e^{-i|k|t+ikx}) &= \int dx\; e^{ikx}(|k|\hat{\phi}(x) - i\hat{\pi}(x)) \\ \hat{a}(e^{-i|k|t+ikx}) &= \int dx\; e^{-ikx}(|k|\hat{\phi}(x) + i\hat{\pi}(x)). \end{aligned} \tag{5.7}$$

Note that we have used the equation

$$(\partial_t\hat{\phi})(0,x) = \hat{\pi}(0,x). \tag{5.8}$$

Equation (5.7) is indeed the standard expressions for the creation and annihilation operators contained in any QFT textbook. (Note that we are using 'Lorentz invariant' normalization for the creation and annihilation operators—the other most common normalization differs by an overall factor of $\sqrt{4\pi|k|}$.)

The good thing about the generalized creation and annihilation operators $\hat{a}^\dagger(f)$ and $\hat{a}(f)$ is that they can be defined for *any* f which solves the wave equation, not just for plane waves. In fact, these creation and annihilation operators satisfy the very nice property

$$[\hat{a}(f), \hat{a}^\dagger(g)] = (f, g)_{\mathrm{KG}}. \tag{5.9}$$

The proof of this formula is as follows:

$$[\hat{a}(f), \hat{a}^\dagger(g)] = -i^2 \int dx\, dy \left[f(x)^*\hat{\pi}(x) - \hat{\phi}(x)\partial_t f(x)^*, g(y)\hat{\pi}(y) - \hat{\phi}(y)\partial_t g(y) \right] \tag{5.10}$$

$$= \int dx\, dy\, \big(f(x)^*\partial_t g(y)\big[\hat{\pi}(x), -\hat{\phi}(y)\big] + \partial_t f^*(x) g(y)\big[-\hat{\phi}(x), \hat{\pi}(y)\big]\big) \tag{5.11}$$

$$= i\int dx \big(f(x)\partial_t g(x) - \partial_t f^*(x) g(x)\big) \tag{5.12}$$

$$= (f, g)_{\mathrm{KG}}. \tag{5.13}$$

As an example, if we compute the commutator of our plane wave creation and annihilation operators, we get the standard relation

$$\begin{aligned} [\hat{a}(e^{-i|k_1|t+ik_1x}), \hat{a}^\dagger(e^{-i|k_2|t+ik_2x})] &= (e^{-i|k_1|t+ik_1x}, e^{-i|k_2|t+ik_2x})_{\mathrm{KG}} \\ &= 4\pi|k_1|\delta(k_1 - k_2). \end{aligned} \tag{5.14}$$

Now, not all solutions to the wave equation f are created equal. We say f is a "positive frequency" solution (with respect to inertial time) if it is a linear combination of solutions of the form $e^{-i|k|t+ikx}$. This will then imply that f^* is a "negative frequency" solution, i.e. it is a linear combination of solutions of the form $e^{i|k|t-ikx}$.

Any introductory QFT textbook will tell you that the standard plane-wave annihilation operators will annihilate the vacuum:

$$\hat{a}(e^{-i|k|t+ikx})\,|0\rangle = 0. \tag{5.15}$$

Therefore, by linearity, we know that for *any* f which is positive frequency, $\hat{a}(f)$ will annihilate the vacuum as well:

$$\hat{a}(f)\,|0\rangle\,, \qquad \text{if } f \text{ is positive frequency.} \tag{5.16}$$

Furthermore, if f is positive frequency, we say $\hat{a}^\dagger(f)\,|0\rangle$ is the single particle state with "wave function" of f.

Notice then that the QM inner product coincides with the KG inner product, because if f and g are both positive frequency,

$$\begin{aligned}\langle 0|\,\hat{a}(f)\,\hat{a}^\dagger(g)\,|0\rangle &= \langle 0|\left([\hat{a}(f),\hat{a}^\dagger(g)] + \hat{a}^\dagger(g)\,\hat{a}(f)\right)|0\rangle \\ &= (f,g)_{\text{KG}}\end{aligned} \tag{5.17}$$

where in the above steps we used (5.9) and (5.16).

Another important property is that the KG inner product of a positive frequency mode and a negative frequency mode is always zero.

$$(f,g)_{\text{KG}} = 0 \qquad \text{if } f \text{ and } g \text{ are positive and negative frequency, respectively.} \tag{5.18}$$

This can be checked by plugging positive and negative frequency plane waves into the KG inner product and calculating that it is 0. The result will extend to all relevant f and g by linearity.

Finally, we note the useful commutation relation is

$$[\hat{\phi}(t,x), \hat{a}^\dagger(f)] = f(t,x). \tag{5.19}$$

The of proof this equation is as follows, where we integrate over the slice at time t:

$$[\hat{\phi}(t,x), \hat{a}^\dagger(f)] = -i\int dy\left[\hat{\phi}(t,x), f(t,y)\hat{\pi}(t,y) - \hat{\phi}(t,y)\partial_t f(t,y)\right] \tag{5.20}$$

$$= f(t,x). \tag{5.21}$$

Equation (5.19) can be used to compute quantities like $\langle 0|\,\hat{\phi}(t,x)\hat{a}^\dagger(f)\,|0\rangle$ as follows:

$$\begin{aligned}\langle 0|\,\hat{\phi}(t,x)\hat{a}^\dagger(f)\,|0\rangle &= \langle 0|\left[\hat{\phi}(t,x), \hat{a}^\dagger(f)\right] + \hat{a}^\dagger(f)\hat{\phi}(t,x)\,|0\rangle \\ &= f(t,x).\end{aligned} \tag{5.22}$$

Note that we also have

$$[\hat{a}(f), \hat{\phi}(t,x)] = f^*(t,x). \tag{5.23}$$

which similarly implies

$$\begin{aligned}\langle 0|\,\hat{a}(f)\hat{\phi}(t,x)\,|0\rangle &= \langle 0|\left[\hat{a}(f), \hat{\phi}(t,x)\right] + \hat{\phi}(t,x)\hat{a}(f)\,|0\rangle \\ &= f^*(t,x).\end{aligned} \tag{5.24}$$

6 Positive and negative frequency as creation and annihilation

From the last section, notice that

$$\hat{a}(f^*) = -\hat{a}^\dagger(f). \tag{6.1}$$

If f is positive frequency (with respect to inertial time t), then f^* is negative frequency. The above equation means that if you try to annihilate a quanta in negative frequency mode, you just end up creating a quanta in a positive frequency mode. The above equation, while it may seem like an abstract piece of mathematical formalism, is in some ways at the heart of why an accelerating observer can detect particles in the vacuum, although it will still take some time to see why that is.

In the meantime, it is worth contemplating the relationship of positive/negative frequency modes to particle creation/annihilation. Let us denote f_+ and f_- to be the positive and negative frequency parts of f, where f is an arbitrary solution to the wave equation.

$$f_+(t,x) \equiv \text{positive inertial frequency part of } f(t,x) \tag{6.2}$$

$$f_-(t,x) \equiv \text{negative inertial frequency part of } f(t,x) \tag{6.3}$$

$f_+(t,x)$ will be a linear combination of positive frequency plane waves of the form $e^{-i|k|t+ikx}$ and $f_-(t,x)$ will be a linear combination of negative frequency plane waves of the form $e^{i|k|t-ikx}$.

Using the above notation, one can clearly see

$$\hat{a}(f) = \hat{a}(f_+ + f_-) = \hat{a}(f_+) - \hat{a}^\dagger(f_-^*) \tag{6.4}$$

$$\hat{a}^\dagger(f) = \hat{a}^\dagger(f_+ + f_-) = \hat{a}^\dagger(f_+) - \hat{a}(f_-^*) \tag{6.5}$$

i.e., if you try to annihilate a mode f, where f contains both a positive frequency mode and a negative frequency mode, you will annihilate the positive frequency mode and create the complex conjugate of the negative frequency mode. (In later sections, when we show that Rindler modes are "mixtures" of positive and negative frequency inertial modes, this will imply that annihilating a Rindler particle simultaneously creates and annihilates inertial particles.)

For clarity later on, let us explicitly define what we mean by positive inertial frequency vs positive Rindler frequency. A mode f which is an eigenfunction of ∂_t has positive inertial frequency if

$$\partial_t f = -i\omega f \quad \text{is positive (negative) } \textbf{inertial} \text{ frequency if } \omega > 0 \;\; (\omega < 0) \tag{6.6}$$

where we also extend the above definitions to linear combinations of such f's.

To define a notion of positive and negative Rindler frequency, we need to remember that the boost Killing vector field $K = K^\mu \partial_\mu$, given in (2.8), points *forwards* in region I, but *backwards* in region II. Because we only ever want to consider notions of positive and negative frequency with respect to observers who are themselves moving forwards in time, we have to modify the definition of "positive Rindler frequency" in region II appropriately. To this end, we define

$$\begin{aligned} Kf = -i\lambda f \quad &\text{is positive (negative) } \textbf{Rindler} \text{ frequency in } \textbf{region I} \text{ if } \;\lambda > 0 \;\; (\lambda < 0) \\ Kf = -i\lambda f \quad &\text{is positive (negative) } \textbf{Rindler} \text{ frequency in } \textbf{region II} \text{ if } \lambda < 0 \;\; (\lambda > 0) \end{aligned} \tag{6.7}$$

where we also extend the above definitions to linear combinations of such f's.

Throughout the rest of these notes, I will try to be clear about the distinction between positive/negative *inertial* frequency and positive/negative *Rindler* frequency. However, if I just use the phrase "positive/negative frequency" and don't specify whether I mean inertial frequency or Rindler frequency, assume I mean inertial frequency.

7 Quantization with Rindler modes

7.1 Rindler creation and annihilation operators

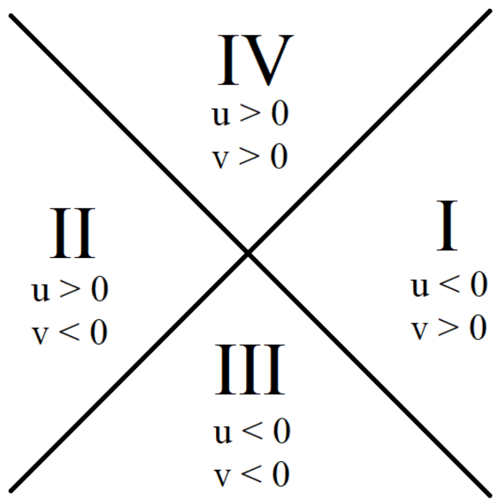

Figure 8: The ranges of u and v on the four regions I, II, III, and IV.

Let us begin by labelling our Rindler modes. We will label them with whether they are left moving or right moving (L or R), their Rindler frequency λ (where $\lambda > 0$), and which quadrants (I, II, III, IV) of spacetime they are non-zero in.

We now give our basis of positive Rindler frequency modes:

$$\begin{aligned}
\Phi^{L,\lambda}_{\mathrm{I,IV}} &= (+v)^{-i|\lambda|}\Theta(+v) \\
\Phi^{L,\lambda}_{\mathrm{II,III}} &= (-v)^{+i|\lambda|}\Theta(-v) \\
\Phi^{R,\lambda}_{\mathrm{I,III}} &= (-u)^{+i|\lambda|}\Theta(-u) \\
\Phi^{R,\lambda}_{\mathrm{II,IV}} &= (+u)^{-i|\lambda|}\Theta(+u).
\end{aligned} \tag{7.1}$$

These are four independent sets of modes which all are all positive Rindler frequency modes. The complex conjugates of the above modes are the negative frequency Rindler modes, making eight sets of basis modes in total. We plot these modes in Figures 9 – 12.

We can likewise define four sets of creation and annihilation operators

$$\begin{aligned}
&\hat{a}^\dagger(\Phi^{L,\lambda}_{\mathrm{I,IV}}), \quad \hat{a}^\dagger(\Phi^{L,\lambda}_{\mathrm{II,III}}), \quad \hat{a}^\dagger(\Phi^{R,\lambda}_{\mathrm{I,III}}), \quad \hat{a}^\dagger(\Phi^{R,\lambda}_{\mathrm{II,IV}}), \\
&\hat{a}(\Phi^{L,\lambda}_{\mathrm{I,IV}}), \quad \hat{a}(\Phi^{L,\lambda}_{\mathrm{II,III}}), \quad \hat{a}(\Phi^{R,\lambda}_{\mathrm{I,III}}), \quad \hat{a}(\Phi^{R,\lambda}_{\mathrm{II,IV}}).
\end{aligned} \tag{7.2}$$

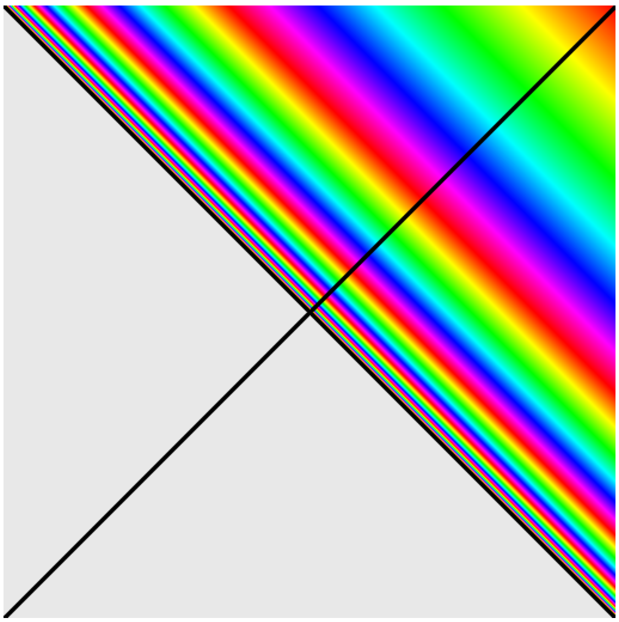

Figure 9: $\Phi^{L,\lambda}_{\mathrm{I,IV}} = \Theta(v)v^{-i|\lambda|}$.

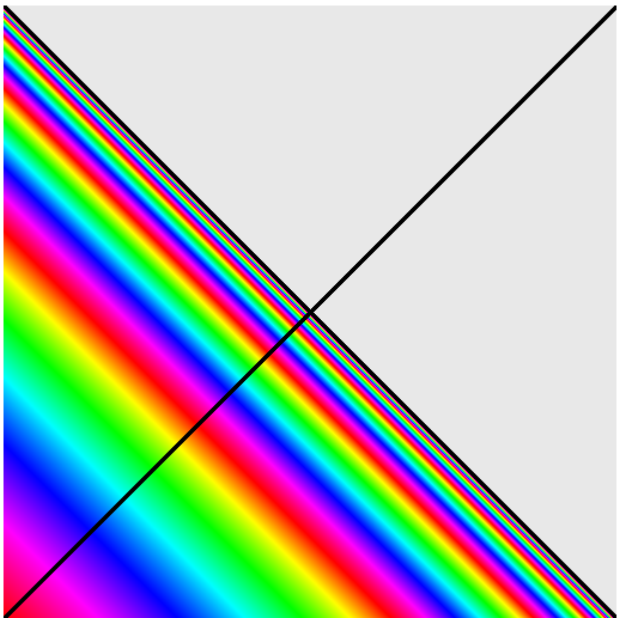

Figure 10: $\Phi^{L,\lambda}_{\mathrm{II,III}} = \Theta(-v)(-v)^{i|\lambda|}$.

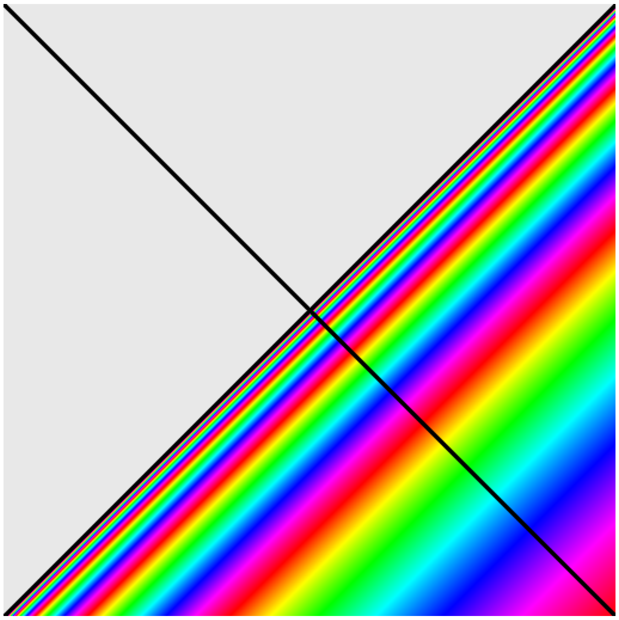

Figure 11: $\Phi^{R,\lambda}_{\mathrm{I,III}} = \Theta(-u)(-u)^{i|\lambda|}$.

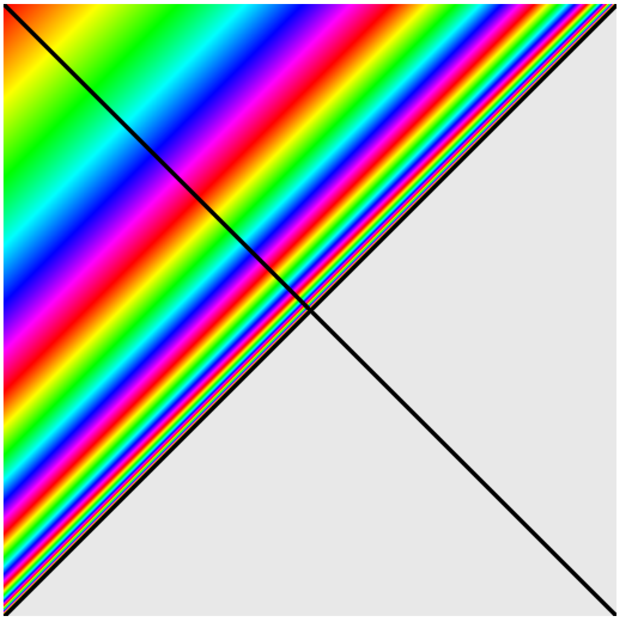

Figure 12: $\Phi^{R,\lambda}_{\mathrm{II,IV}} = \Theta(u)u^{-i|\lambda|}$.

In order to find the commutation relations of the creation and annihilation operators, then by (5.9) we just need to compute the KG inner products of the modes in (7.1). For instance, we can compute on the $t = 0$ slice

$$(\Phi^{L,\lambda_1}_{\text{I,IV}}, \Phi^{L,\lambda_2}_{\text{I,IV}})_{\text{KG}} = i\int_{-\infty}^{\infty} dx\big((\Phi^{L,\lambda_1}_{\text{I,IV}})^*\partial_t\Phi^{L,\lambda_2}_{\text{I,IV}} - \Phi^{L,\lambda_2}_{\text{I,IV}}\partial_t(\Phi^{L,\lambda_1}_{\text{I,IV}})^*\big) \tag{7.3}$$

$$= i\int_{0}^{\infty} dx\big(-i\lambda_2 x^{i\lambda_1}x^{-i\lambda_2-1} - i\lambda_1 x^{-i\lambda_2}x^{i\lambda_1-1}\big) \tag{7.4}$$

$$= (\lambda_1+\lambda_2)\int_0^\infty \frac{dx}{x}x^{i(\lambda_1-\lambda_2)} \tag{7.5}$$

$$= (\lambda_1+\lambda_2)\int_{-\infty}^\infty dk\; e^{ik(\lambda_1-\lambda_2)} \tag{7.6}$$

$$= 4\pi\lambda_1\delta(\lambda_1-\lambda_2). \tag{7.7}$$

The other KG inner products follow in a similar manner. Furthermore, the KG inner product of an R mode and an L mode is 0, and the KG inner product of a I mode and a II mode is also 0.

We therefore have the commutators

$$\begin{aligned}
[\hat{a}(\Phi^{L,\lambda_1}_{\text{I,IV}}), \hat{a}^\dagger(\Phi^{L,\lambda_2}_{\text{I,IV}})] &= 4\pi\lambda_1\delta(\lambda_1-\lambda_2),\\
[\hat{a}(\Phi^{L,\lambda_1}_{\text{II,III}}), \hat{a}^\dagger(\Phi^{L,\lambda_2}_{\text{II,III}})] &= 4\pi\lambda_1\delta(\lambda_1-\lambda_2),\\
[\hat{a}(\Phi^{R,\lambda_1}_{\text{I,III}}), \hat{a}^\dagger(\Phi^{R,\lambda_2}_{\text{I,III}})] &= 4\pi\lambda_1\delta(\lambda_1-\lambda_2),\\
[\hat{a}(\Phi^{R,\lambda_1}_{\text{II,IV}}), \hat{a}^\dagger(\Phi^{R,\lambda_2}_{\text{II,IV}})] &= 4\pi\lambda_1\delta(\lambda_1-\lambda_2),
\end{aligned} \tag{7.8}$$

where all other commutators vanish.

In QFT, the Minkowski vacuum state of an inertial observer is usually denoted by $|0\rangle$, but for clarity we shall denote it here by $|0_M\rangle$. Likewise, we will denote a separate state $|0_R\rangle$ as the "Rindler vacuum." This state has no particles according to an accelerating observer. It satisfies

$$\hat{a}^\dagger(\Phi^{L,\lambda}_{\text{I,IV}})\,|0_R\rangle = \hat{a}^\dagger(\Phi^{L,\lambda}_{\text{II,III}})\,|0_R\rangle = \hat{a}^\dagger(\Phi^{R,\lambda}_{\text{I,III}})\,|0_R\rangle = \hat{a}^\dagger(\Phi^{R,\lambda}_{\text{II,IV}})\,|0_R\rangle = 0. \tag{7.9}$$

Because the basis of Rindler modes is complete, the above equation uniquely specifies the state $|0_R\rangle$. We will soon see that $|0_M\rangle \neq |0_R\rangle$.

7.2 The Trick™ to solve for $|0_M\rangle$ using complex analysis

There is a very clever trick which we can use to solve for $|0_M\rangle$ in terms of $|0_R\rangle$. The trick involves investigating the particular wave functions $(v - i\epsilon)^{-i\lambda}$ and $(u - i\epsilon)^{-i\lambda}$ where ϵ is a tiny positive constant which is to be taken to 0. It turns out that these wave functions are both positive inertial frequency functions. That is to say, both $(v - i\epsilon)^{-i\lambda}$ and $(u - i\epsilon)^{-i\lambda}$ can be written as linear combinations of positive frequency plane waves $e^{-i|k|t+ikx}$. This can be seen using complex analysis. Setting $x = 0$, both functions become $f(t) = (t - i\epsilon)^{-i\lambda}$. Let us now view t as a complex coordinate. Notice that $(t - i\epsilon)^{-i\lambda}$ has a branch cut in the upper half plane, but is bounded and holomorphic in the lower half plane. If one were to compute the Fourier transform $\widetilde{f}(\omega)$ defined by

$$\widetilde{f}(\omega) = \int dt\; e^{i\omega t} f(t) \tag{7.10}$$

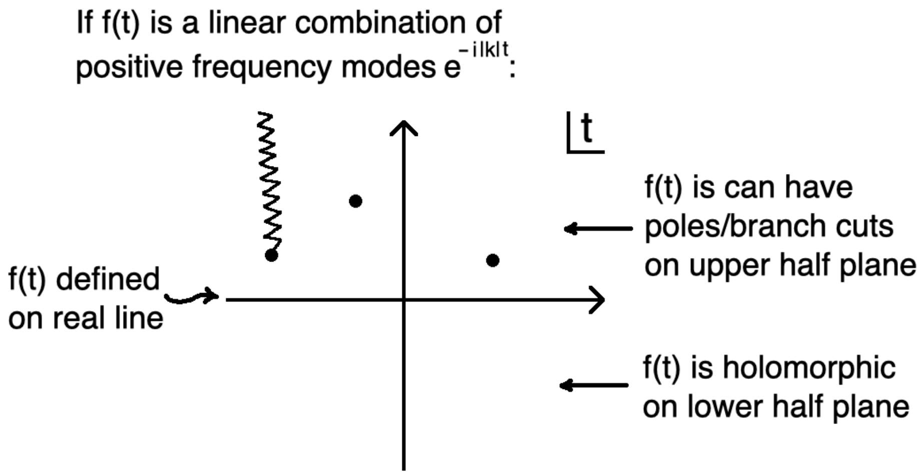

Figure 13: A function of t which is a linear combination of positive frequency exponentials $e^{-i|k|t}$ will be holomorphic in the lower half plane, but may have poles and branch cuts in the upper half plane.

then, because the exponential decays in the lower half t-plane for $\omega < 0$, we could close the contour and find $\widetilde{f}(\omega) = 0$ for $\omega < 0$. Therefore, $f(t)$ can be written purely as a sum of positive frequency modes $e^{-i|k|t}$.

Indeed, it is a well known mathematical fact[1] that any (reasonable) function $f(t)$ which is a linear combination of positive frequency waves $e^{-i|k|t}$ must be holomorphic (i.e., have no poles or branch cuts) in the lower half plane. This is depicted in Figure 13.

Because $(v - i\epsilon)^{-i\lambda}$ and $(u - i\epsilon)^{-i\lambda}$ are positive inertial frequency modes, we know that annihilation operators constructed out of these modes must annihilate the Minkowski vacuum:

$$\hat{a}((v - i\epsilon)^{-i\lambda}) |0_M\rangle = \hat{a}((u - i\epsilon)^{-i\lambda}) |0_M\rangle = 0. \tag{7.11}$$

However, these functions can also be written as sums of the Rindler modes in (7.1), via the equations (where we assume $\lambda > 0$):

$$(v - i\epsilon)^{-i\lambda} = \Theta(v) v^{-i\lambda} + e^{-\pi\lambda} \Theta(-v)(-v)^{-i\lambda} = \Phi^{L,\lambda}_{\text{I,IV}} + e^{-\pi\lambda} (\Phi^{L,\lambda}_{\text{II,III}})^* \tag{7.12}$$

$$(u - i\epsilon)^{-i\lambda} = \Theta(u) u^{-i\lambda} + e^{-\pi\lambda} \Theta(-u)(-u)^{-i\lambda} = \Phi^{R,\lambda}_{\text{II,IV}} + e^{-\pi\lambda} (\Phi^{R,\lambda}_{\text{I,III}})^*. \tag{7.13}$$

In order to check the above equations hold for negative values of v and u, one must use the $i\epsilon$ in order to navigate around the origin. This is because whenever a negative number is raised to an imaginary power, there is an inherent ambiguity that must be addressed. In evaluating $(-1)^i$, do we use $-1 = e^{i\pi}$ or $-1 = e^{-i\pi}$? The first option gives us $(-1)^i = e^{-\pi}$ while the second gives $(-1)^i = e^{\pi}$. Thankfully, when it comes to evaluating $(v - i\epsilon)^{-i\lambda}$ for negative v, the $i\epsilon$ takes care of that ambiguity for us: it instructs us to write $-1 = e^{-i\pi}$ in order to avoid the singularity at $v = +i\epsilon$. This is drawn in (14).

For completeness, we will also write down the corresponding sums with the negative frequency

[1] See for instance Theorem 19.2 of Paley and Weiner in [4] for a more rigorous statement of the fact.

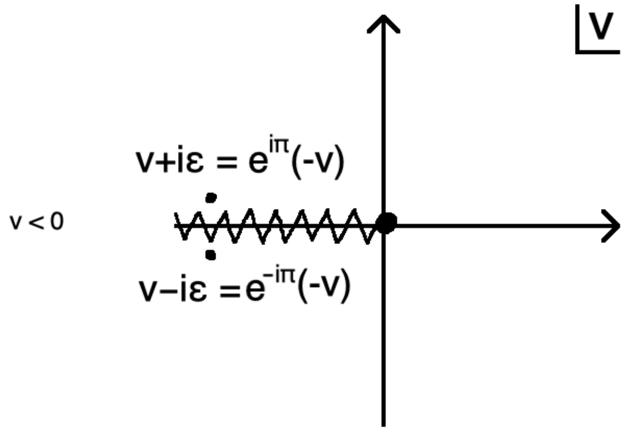

Figure 14: If one wants to calculate $(v - i\epsilon)^{-i\lambda}$ for negative values of v, one must go beneath the branch cut, circling the origin by a phase $e^{-i\pi}$. If one wants to calculate $(v + i\epsilon)^{-i\lambda}$, one must go above the branch cut, circling the origin by a phase $e^{i\pi}$.

modes $(v + i\epsilon)^{-i\lambda}$ and $(u + i\epsilon)^{-i\lambda}$ (where we assume $\lambda > 0$):

$$(v + i\epsilon)^{-i\lambda} = \Theta(v) v^{-i\lambda} + e^{+\pi\lambda}\Theta(-v)(-v)^{-i\lambda} = \Phi^{L,\lambda}_{\text{I,IV}} + e^{+\pi\lambda}(\Phi^{L,\lambda}_{\text{II,III}})^* \tag{7.14}$$

$$(u + i\epsilon)^{-i\lambda} = \Theta(u) u^{-i\lambda} + e^{+\pi\lambda}\Theta(-u)(-u)^{-i\lambda} = \Phi^{R,\lambda}_{\text{II,IV}} + e^{+\pi\lambda}(\Phi^{R,\lambda}_{\text{I,III}})^* \tag{7.15}$$

In any case, we can see that the Rindler modes in (7.1) are *sums* of positive and negative inertial frequency modes. We will have more to say about this in the next section.

From the linearity of the KG inner product, equations (7.12), (7.13), and (6.1), we instantly have

$$\begin{aligned} \hat{a}((v - i\epsilon)^{-i\lambda}) &= \hat{a}(\Phi^{L,\lambda}_{\text{I,IV}}) - e^{-\pi\lambda}\hat{a}^\dagger(\Phi^{L,\lambda}_{\text{II,III}}) \\ \hat{a}((u - i\epsilon)^{-i\lambda}) &= \hat{a}(\Phi^{R,\lambda}_{\text{II,IV}}) - e^{-\pi\lambda}\hat{a}^\dagger(\Phi^{R,\lambda}_{\text{I,III}}). \end{aligned} \tag{7.16}$$

From (7.11), we then have the equations

$$\begin{aligned} \hat{a}(\Phi^{L,\lambda}_{\text{I,IV}})\left|0_M\right\rangle &= e^{-\pi\lambda}\hat{a}^\dagger(\Phi^{L,\lambda}_{\text{II,III}})\left|0_M\right\rangle \\ \hat{a}(\Phi^{R,\lambda}_{\text{II,IV}})\left|0_M\right\rangle &= e^{-\pi\lambda}\hat{a}^\dagger(\Phi^{R,\lambda}_{\text{I,III}})\left|0_M\right\rangle \end{aligned} \tag{7.17}$$

We are about to us the above two equations to write $\left|0_M\right\rangle$ using Rindler modes—but before we do that, let us use a more convenient normalization convention for our creation and annihilation operators. Defining

$$\begin{aligned} \hat{a}^\dagger_{L,\text{I},\lambda} &= \sqrt{\frac{1}{4\pi\lambda\delta(0)}}\hat{a}^\dagger(\Phi^{L,\lambda}_{\text{I,IV}}) \qquad & \hat{a}^\dagger_{L,\text{II},\lambda} &= \sqrt{\frac{1}{4\pi\lambda\delta(0)}}\hat{a}^\dagger(\Phi^{L,\lambda}_{\text{II,III}}) \\ \hat{a}^\dagger_{R,\text{I},\lambda} &= \sqrt{\frac{1}{4\pi\lambda\delta(0)}}\hat{a}^\dagger(\Phi^{R,\lambda}_{\text{I,III}}) \qquad & \hat{a}^\dagger_{R,\text{II},\lambda} &= \sqrt{\frac{1}{4\pi\lambda\delta(0)}}\hat{a}^\dagger(\Phi^{R,\lambda}_{\text{II,IV}}) \end{aligned} \tag{7.18}$$

with the non-zero commutators

$$\begin{aligned} [\hat{a}_{R,\text{I},\lambda}, \hat{a}^\dagger_{R,\text{I},\lambda}] = 1, \quad [\hat{a}_{R,\text{II},\lambda}, \hat{a}^\dagger_{R,\text{II},\lambda}] = 1, \\ [\hat{a}_{L,\text{I},\lambda}, \hat{a}^\dagger_{L,\text{I},\lambda}] = 1, \quad [\hat{a}_{L,\text{II},\lambda}, \hat{a}^\dagger_{L,\text{II},\lambda}] = 1. \end{aligned} \tag{7.19}$$

We then rewrite (7.17) with the newly normalized operators as

$$\begin{aligned}\hat{a}_{L,\mathrm{I},\lambda}\left|0_M\right\rangle &= e^{-\pi\lambda}\hat{a}^\dagger_{L,\mathrm{II},\lambda}\left|0_M\right\rangle ,\\ \hat{a}_{R,\mathrm{II},\lambda}\left|0_M\right\rangle &= e^{-\pi\lambda}\hat{a}^\dagger_{R,\mathrm{I},\lambda}\left|0_M\right\rangle ,\end{aligned} \tag{7.20}$$

which is solved by[2]

$$\left|0_M\right\rangle = \prod_{\lambda>0} N_\lambda \exp\Big(e^{-\pi\lambda}\hat{a}^\dagger_{R,\mathrm{I},\lambda}\hat{a}^\dagger_{R,\mathrm{II},\lambda}\Big) N_\lambda \exp\Big(e^{-\pi\lambda}\hat{a}^\dagger_{L,\mathrm{I},\lambda}\hat{a}^\dagger_{L,\mathrm{II},\lambda}\Big)\left|0_R\right\rangle \tag{7.21}$$

In order to show this, all one has to do is use the equation[3]

$$\left[\hat{a}_{L,\mathrm{I},\lambda}, \exp\left(\sum_{\lambda>0} e^{-\pi\lambda}\hat{a}^\dagger_{L,\mathrm{I},\lambda}\hat{a}^\dagger_{L,\mathrm{II},\lambda}\right)\right] = e^{-\pi\lambda}\hat{a}^\dagger_{L,\mathrm{II},\lambda}\exp\left(\sum_{\lambda>0} e^{-\pi\lambda}\hat{a}^\dagger_{L,\mathrm{I},\lambda}\hat{a}^\dagger_{L,\mathrm{II},\lambda}\right) \tag{7.22}$$

as well as the analogous equation for the right-moving modes. Note that we have defined the normalization constants

$$N_\lambda = \frac{1}{\sqrt{1-e^{-2\pi\lambda}}} \tag{7.23}$$

which ensures that $\langle 0_M|0_M\rangle = 1$.

We now rewrite (7.21) as

$$\left|0_M\right\rangle = \prod_\lambda \left(N_\lambda \sum_{n=0}^{\infty} e^{-n\pi\lambda}\left|n,\mathrm{I}\right\rangle_L \otimes \left|n,\mathrm{II}\right\rangle_L\right) \otimes \left(N_\lambda \sum_{m=0}^{\infty} e^{-m\pi\lambda}\left|m,\mathrm{I}\right\rangle_R \otimes \left|m,\mathrm{II}\right\rangle_R\right) \tag{7.24}$$

where $\left|n,\mathrm{I}\right\rangle_L$, $\left|n,\mathrm{II}\right\rangle_L$, $\left|n,\mathrm{I}\right\rangle_R$, $\left|n,\mathrm{II}\right\rangle_R$ are the states on the subsectors of the Hilbert space

$$\mathcal{H} = \mathcal{H}_{L,\mathrm{I}} \otimes \mathcal{H}_{L,\mathrm{II}} \otimes \mathcal{H}_{R,\mathrm{I}} \otimes \mathcal{H}_{R,\mathrm{II}} \tag{7.25}$$

of left-moving/right-moving Rindler modes in regions I/II with occupation number n.

If we trace out region II from the state $\left|0_M\right\rangle$, the density matrix of the state becomes

$$\begin{aligned}\rho_\mathrm{I} &= \mathrm{Tr}_\mathrm{II}(\left|0_M\right\rangle\left\langle 0_M\right|)\\ &= \prod_\lambda \left(N_\lambda^2 \sum_{n=0}^{\infty} e^{-n2\pi\lambda}\left|n,I\right\rangle_R \left\langle n,I\right|_R\right)\left(N_\lambda^2 \sum_{m=0}^{\infty} e^{-m2\pi\lambda}\left|m,I\right\rangle_L \left\langle m,I\right|_L\right)\\ &= \frac{e^{-2\pi\hat{K}_\mathrm{I}}}{\mathrm{Tr}_\mathrm{I}(e^{-2\pi\hat{K}_\mathrm{I}})}\end{aligned} \tag{7.26}$$

where $\hat{K}_\mathrm{I}$ is the self-adjoint Hilbert space boost generator restricted to region I, i.e. if $\hat{K}$ is the boost generator on the full Hilbert space, then $\hat{K} = \hat{K}_\mathrm{I} + \hat{K}_{II}$. See appendix G for more details on the boost operator.

[2]If you had some doubts that $\left|0_R\right\rangle$ as defined by (7.9) really existed, you can simply invert (7.21) to express $\left|0_R\right\rangle$ using oscillator excitations on top of $\left|0_M\right\rangle$.

[3]This equation can be easily checked with the following observation. Schematically, creation and annihilation operators have the commutation relation $[\hat{a},\hat{a}^\dagger] = 1$. This is the same commutation relation as $[\frac{\partial}{\partial x}, x] = 1$, meaning we can regard $\hat{a}$ as $\frac{\partial}{\partial \hat{a}^\dagger}$ and use the identity $[\frac{\partial}{\partial x}, f(x)] = f'(x)$.

Now, because the boost vector field K aligns with the trajectory accelerating trajectory, the boost operator $\hat{K}$ gives the natural notion of time evolution used by the Rindler observer. However, we must also account for the normalization. In Minkowski coordinates, the vector field has components $K^\mu = (x, t)$. On the accelerated trajectory, it has the norm $K^\mu K_\mu = a^{-2}$. Therefore, it is actually the operator $a\hat{K}$ which corresponds to proper time evolution along the accelerating observer's worldline. We therefore write

$$\rho_{\mathrm{I}} = \frac{e^{-\frac{2\pi}{a}(a\hat{K}_{\mathrm{I}})}}{\mathrm{Tr}_{\mathrm{I}}(e^{-\frac{2\pi}{a}(a\hat{K}_{\mathrm{I}})})} \tag{7.27}$$

and note that the density matrix has exactly the form of a thermal density matrix, with $\beta = \frac{2\pi}{a}$! Therefore, the accelerating observer will see themselves "bathed" in a thermal bath of particles at the temperature $a/2\pi$!

Having said that, the above discussion may come off as a bit abstract. In the later sections, we will explicitly model a particle detector and see that it really would detect this thermal bath of particles when travelling on a uniformly accelerated worldline.

8 Mixing of positive and negative frequency modes

In the last section we noted that modes which are positive frequency in inertial time are mixtures of positive and negative frequency modes in Rindler time. Likewise, modes which are positive frequency in Rindler time are mixture of positive and negative frequency modes in inertial time. (This was shown in equations (7.12)–(7.15).)

À la equation (7.16), the mixture of positive and negative frequency modes has profound implications upon quantization: it means that *annihilating* an inertial particle corresponds to a superposition of *creating and annihilating* Rindler particles in different halves of the spacetime. We then showed that this implied that the Rindler observer would see the Minkowski vacuum as containing a thermal bath of particles.

It is therefore worth meditating further on this mixing of positive and negative frequency modes. Aside from the clever complex analysis trick given in the previous section, there are two more ways to see that the positive and negative frequency modes *must* mix.

Let us describe the first reason. There is a mathematical theorem [5] which states that if a function $f(x)$ has a Fourier transform which only runs over $k > 0$

$$f(x) = \int_0^\infty dk\, \widetilde{f}(k) e^{-ikx} \tag{8.1}$$

then if $f(x)$ vanishes for any finite interval of $x \in \mathbb{R}$, then it must vanish everywhere. This is because $f(x)$ must be holomorphic on the lower half plane, and if a holomorphic function has a dense set of limit points where it tends to 0, it must be identically 0. (I warn the reader that making this statement and proof completely rigorous is very demanding [6].)

Therefore, if $f(x)$ is 0 on the halfline $x < 0$, then it must be a linear combination of plane waves e^{-ikx} for both positive and negative k.

If we then let our function depend on time as well as $f(t, x)$ and demand it solves the wave equation, and without loss of generality assume it is a left-moving function, i.e. be a function

$f(t+x)$, then the restriction that it must reduce at $t=0$ to a sum of plane waves like e^{-ikx} with both $k>0$ and $k<0$ requires that it must also be a sum of positive and negative frequency modes $e^{-i|k|t}$ and $e^{i|k|t}$ in time t.

Therefore, we can see that the any solution to the wave equation which vanishes in some finite interval of space on (like our Rindler modes do) must be a sum of both positive and negative frequency modes. This concludes our the first reason.

Let us now describe the second reason. It essentially boils down to the fact that the boost vector field K points in different directions in regions I and II, although the details are a bit more involved.

Imagine you wanted to construct a solution to the wave equation which was an eigenfuction of the boost vector field K. How would you do it? One way would be to take a plane wave and construct a "weighted average" over all boosts, which would make it a boost eigenstate by construction. More concretely, say you started with the plane wave $e^{-i|\kappa|t+\kappa x}$ for some κ. If we define a function F which is our "weighted average" over all boosts,

$$F(t,x)=\int_{-\infty}^{\infty} d\theta\ e^{i\lambda\theta}e^{\theta K}e^{-i|\kappa|t+\kappa x} \tag{8.2}$$

then we can see that F is an eigenfunction of K with eigenvalue $-i\lambda$ through the simple algebra

$$\begin{aligned} KF &= \int_{-\infty}^{\infty} d\theta e^{i\lambda\theta}Ke^{\theta K}e^{-i|\kappa|t+\kappa x} \\ &= \int_{-\infty}^{\infty} d\theta e^{i\lambda\theta}\frac{d}{d\theta}e^{\theta K}e^{-i|\kappa|t+\kappa x} \\ &= -\int_{-\infty}^{\infty} d\theta\frac{d}{d\theta}e^{i\lambda\theta}e^{\theta K}e^{-i|\kappa|t+\kappa x} \\ &= -i\lambda F. \end{aligned} \tag{8.3}$$

Now, from its definition (8.2), F is manifestly positive frequency with respect to inertial time. Furthermore, if we assume $\lambda>0$, then from the equation $KF=-i\lambda F$, we can clearly see that F has positive Rindler frequency in region I and negative Rindler frequency in region II via (6.7). Therefore, from this averaging procedure, we have shown how to construct functions which have positive inertial frequency but are mixtures of positive and negative Rindler frequency.

Let us now evaluate F. Without loss of much generality, we take $\kappa>0$ and also introduce a dampening factor $t\to t-i\epsilon$ into the integral to make it converge. Using the equation $e^{\theta K}f(v)=f(e^{\theta}v)$, we compute

$$\begin{aligned} F &= \int_{-\infty}^{\infty} d\theta e^{i\lambda\theta}e^{-i\kappa e^{\theta}(v-i\epsilon)} \\ &= \int_{0}^{\infty}\frac{dk}{k}k^{i\lambda}e^{-i\kappa k(v-i\epsilon)} \\ &= (v-i\epsilon)^{-i\lambda}\kappa^{-i\lambda}e^{\pi\lambda/2}\Gamma(i\lambda) \end{aligned} \tag{8.4}$$

and confirm that F is proportional to our function $(v-i\epsilon)^{-i\lambda}$ from before.

9 Modelling a Particle Detector

9.1 The Unruh-DeWitt Detector Hamiltonian

In this section we will explain how to model a simple "particle detector." Imagine we have an observer that moves along some worldline $(t, x(t))$. The observer holds a detector which has a two-level Hilbert space of $\{|\downarrow\rangle, |\uparrow\rangle\}$. The $|\downarrow\rangle$ state has an energy of 0 and the $|\uparrow\rangle$ state has an energy of E. The observer begins the measurement with the detector in the $|\downarrow\rangle$ state. If there is a particle in the spacetime that crosses the observer's path, there is some probability that the detector will absorb the particle, transition to the $|\uparrow\rangle$ state, and go "bing!".

The total Hamiltonian of the scalar field and detector is

$$H = H_0 + W(t) \tag{9.1}$$

$$H_0 = H_0^{\text{QFT}} + H_0^{\text{d}} \tag{9.2}$$

$$H_0^{\text{d}} = E\,\gamma^{-1}\,|\uparrow\rangle\langle\uparrow| \tag{9.3}$$

$$W(t) = g\,\gamma^{-1}\,w(t)\big(\,|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\,\big)\hat{\phi}(x(t)) \tag{9.4}$$

where H_0^{QFT} is the free hamiltonian of the scalar field given in (4.2), H_0^{d} is the free Hamiltonian of the detector, $\gamma = dt/d\tau = (1 - \dot{x}^2)^{-1/2}$ is the time dilation factor from the motion of the observer,[4] and $W(t)$ is the interaction term between the detector and the quantum field. g is a small coupling constant and $w(t)$ is a window function which is 0 when the measurement device is off and 1 when the measurement device is on. Because ϕ is dimensionless in $1 + 1$ dimensions, g must have the dimensions of energy.

9.2 Scenario 1: inertial observer detecting a single particle

Say that the initial state of the quantum field has exactly one particle in it with positive frequency wave function ψ. The initial joint state $|i\rangle$ of the quantum field + detector is therefore

$$|i\rangle = \hat{a}^\dagger(\psi)\,|0, \downarrow\rangle\,. \tag{9.5}$$

We are interested in computing the probability amplitude that we end up in the final state $|f\rangle$ where the quantum field is in the vacuum and the detector has transitioned to the $|\uparrow\rangle$ state

$$|f\rangle = |0, \uparrow\rangle\,, \tag{9.6}$$

i.e., that the particle is absorbed and the detector goes "bing!"

To calculate the probability amplitude $c_{i\to f}$ to leading order in the small coupling constant g, we shall use Fermi's golden rule, which is reviewed and proven in Appendix D. We reproduce (D.7) here:

$$c_{i\to f} = -i\int_{t_i}^{t_f} dt\;\,\langle f|\,e^{iH_0 t}W(t)e^{-iH_0 t}\,|i\rangle\,. \tag{9.7}$$

[4] Technically, γ and H_0^{d} are time dependant because $\dot{x}(t)$ is time dependant. However, this can be accounted for simply by using the observer's proper time τ, making the detector's phase evolve as $e^{-iE\tau}$ and be effectively time independent.

Plugging in our definitions of $|i\rangle$, $|f\rangle$, H_0 and $W(t)$, the above equation reduces to

$$c_{i\to f} = -ig \int \frac{dt}{\gamma}\; w(t) e^{iE\tau(t)} \langle 0|\, e^{itH_0^{\text{QFT}}} \hat{\phi}(x(t)) e^{-itH_0^{\text{QFT}}} a^\dagger(\psi)\, |0\rangle \tag{9.8}$$

$$= -ig \int d\tau\; w(\tau) e^{iE\tau} \langle 0|\, \hat{\phi}(t(\tau), x(\tau)) \hat{a}^\dagger(\psi)\, |0\rangle \tag{9.9}$$

$$= -ig \int d\tau\; w(\tau) e^{iE\tau} \psi(t(\tau), x(\tau)). \tag{9.10}$$

where τ is the proper time of the observer along the worldline, and in the final step we used equation (5.22).

Armed with the above expression, let's see if our detector really works as advertised.

Say that our our observer's worldline just sits at the middle of the spacetime

$$(t(\tau), x(\tau)) = (\tau, 0), \tag{9.11}$$

and the detector turns on at time $t = -T/2$ and stays on until $t = T/2$:

$$w(t) = \begin{cases} 1 & \text{if } -\frac{T}{2} < t < \frac{T}{2} \\ 0 & \text{if otherwise} \end{cases}. \tag{9.12}$$

Furthermore, let's say that the particle's wave function is a plane wave:

$$\psi(t, x) = e^{-i|k|t + ikx}. \tag{9.13}$$

If we assume T is large, then

$$\begin{aligned} c_{i\to f} &= -ig \int_{-T/2}^{T/2} d\tau\; e^{iE\tau} e^{-i|k|\tau} \\ &= -ig(2\pi)\delta(E - |k|) \end{aligned} \tag{9.14}$$

and the detector will only absorb a particle if the particle's energy equals E.

If we want to turn the amplitude $c_{i\to f}$ into a probability $P_{i\to f}$, we have to be careful about the normalization of our initial state $\hat{a}^\dagger(\psi)\,|0\rangle$, because plane waves have a distributional Dirac delta function normalization:

$$\begin{aligned} \langle 0|\, a^\dagger(e^{-i|k_1|t + ik_1x}) a(e^{-i|k_2|t+ik_2x})\, |0\rangle &= (e^{-i|k_1|t+ik_1x}, e^{-i|k_2|t+ik_2x})_{\text{KG}} \\ &= (2\pi)2|k_1|\delta(k_1 - k_2). \end{aligned} \tag{9.15}$$

Dividing by this infinite factor, we then have

$$\begin{aligned} P_{i\to f} &= \frac{|c_{i\to f}|^2}{\langle 0|\, \hat{a}(\psi)\hat{a}^\dagger(\psi)\, |0\rangle} \\ &= \frac{g^2(2\pi)^2\delta(0)\delta(E - |k|)}{(2\pi)2|k|\delta(|k| - |k|)} \\ &= \frac{g^2T}{2E}\delta_{E,|k|} \end{aligned} \tag{9.16}$$

where in the last line we used the replacement rule $2\pi\delta(0) \to T$ (this replacement is justified more rigorously in Appendix C) and also used $\frac{\delta(E-|k|)}{\delta(|k|-|k|)} = \delta_{E,|k|}$ where $\delta_{E,|k|}$ is a continuous version of the Kronecker delta which is 1 when $E = |k|$ and 0 otherwise.

We therefore see that our particle detector really does behave as it should: it will absorb a particle of energy E with probability $\frac{g^2 T}{2E}$!

Before moving on, let's review all of the approximations necessary to make this analysis valid. In order to use Fermi's golden rule consistent to the first order in perturbation theory, we can only trust this approximation when

$$P_{i\to f} = \frac{g^2 T}{2E} \ll 1 \tag{9.17}$$

so that our transition amplitude never grows close to 1, at which point our approximation would break down.

Furthermore, in order to replace the oscillatory integral by a Dirac delta function in (9.14), we needed to use

$$(E - |k|)T \gg 1. \tag{9.18}$$

Taken together, these two approximations mean that g is small and T is large, but g is parametrically smaller than T is large, i.e.

$$gT \ll 1. \tag{9.19}$$

In the following sections we will work in the limit $T \to \infty$ for simplicity, but in the back of our minds will we will remember that T is technically finite and g must be extremely small.

9.3 Scenario 2: inertial observer in thermal bath

Let us now repeat the analysis of the last section with one small modification: what if our stationary observer was bathed in a thermal bath of particles?

One can show that if there are n particles in the spacetime instead of just one, then the probability of detection $P_{i\to f}$ is multiplied by n.

So, what if the probability of there being n particles of energy E in a certain mode were $\sim e^{-n\beta E}$? Defining the partition function as

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-n\beta E} = \frac{1}{1 - e^{-\beta E}} \tag{9.20}$$

then the probability of detection when there are n particles in a mode is

$$P(n) = \left(\frac{n e^{-n\beta E}}{Z(\beta)} \right) \frac{g^2 T}{2E}. \tag{9.21}$$

To find the total probability of detecting a particle, we must sum over all n and multiply by $\times 2$. The multiplication by 2 is necessary because there are two modes with energy E, corresponding to both right-moving and left-moving plane waves.

$$\begin{aligned}P_{\text{tot}} &= 2\sum_{n=0}^{\infty} P(n) = \frac{g^2 T}{E}\frac{1}{Z(\beta)}\sum_{n=0}^{\infty} n e^{-n\beta E} = \frac{g^2 T}{E}\frac{1}{Z(\beta)}\frac{-\partial_\beta}{E}Z(\beta) \\ &= \frac{g^2 T}{E}\frac{e^{-\beta E}}{1-e^{-\beta E}}\end{aligned} \tag{9.22}$$

We shall keep the above result in mind for the next section, because when we calculate the probability that a Rindler observer's detector goes "bing!" in the vacuum, we will find it is the exact same probability!

9.4 Scenario 3: Rindler observer in the vacuum

Now we come to the main event: what if we have a Rindler observer who is accelerating in the vacuum? With what probability will their detector go "bing!"?

The worldline of the observer is

$$(t(\tau), x(\tau)) = \big(\frac{1}{a}\sinh(a\tau), \frac{1}{a}\cosh(a\tau)\big) \tag{9.23}$$

and for mathematical simplicity, we say the detector is on at all times:

$$w(t) = 1. \tag{9.24}$$

Because we want to calculate the total probability P_{tot} that the detector goes "bing!" for all possible final quantum field states, we must take Fermi's golden rule (9.7) and sum it over all $|f\rangle$. From Appendix D.2, the result is

$$P_{\text{tot}} = \int_{t_i}^{t_f} dt' \int_{t_i}^{t_f} dt \ \langle i| \, e^{iH_0 t'} W(t') e^{-iH_0(t'-t)} W(t) e^{-iH_0 t} \, |i\rangle \tag{9.25}$$

If we set the initial state to be

$$|i\rangle = |0, \downarrow\rangle \tag{9.26}$$

then

$$P_{\text{tot}} = g^2 \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \ e^{-iE(\tau_1-\tau_2)} \langle 0| \, \hat{\phi}\big(t(\tau_1), x(\tau_1)\big) \, \hat{\phi}\big(t(\tau_2), x(\tau_2)\big) \, |0\rangle \,. \tag{9.27}$$

Using boost-time translational invariance, we can use the variable

$$\tau = \tau_1 - \tau_2 \tag{9.28}$$

and we have

$$P_{\text{tot}} = g^2 \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau \, \langle 0| \, \hat{\phi}\big(t(\tau), x(\tau)\big) \, \hat{\phi}\big(t(0), x(0)\big) \, |0\rangle \,. \tag{9.29}$$

Regarding the infinite $\int_{-\infty}^{\infty} d\tau_1$ factor as the total measurement time T, the transition rate per unit time is

$$\frac{d}{d\tau}P_{\text{tot}} = g^2 \int_{-\infty}^{\infty} d\tau \, \langle 0| \, \hat{\phi}\big(t(\tau), x(\tau)\big) \, \hat{\phi}\big(t(0), x(0)\big) \, |0\rangle \, . \tag{9.30}$$

Therefore, the transition rate is determined completely by the non-time-ordered two point function. In Appendix E, this two point function is calculated to be

$$\langle 0| \, \hat{\phi}(t_1, x_1)\hat{\phi}(t_2, x_2) \, |0\rangle = -\frac{1}{4\pi} \log \Big[(t_1 - t_2)^2 - (x_1 - x_2)^2 - i\epsilon \, \text{sgn}(t_1 - t_2) \Big]. \tag{9.31}$$

We note that

$$(t(\tau) - t(0))^2 - (x(\tau) - x(0))^2 = \frac{4}{a^2} \sinh^2(a\tau/2) \tag{9.32}$$

and, because $\text{sgn}(t(\tau) - t(0)) = \text{sgn}(\tau)$, we have

$$(t(\tau) - t(0))^2 - (x(\tau) - x(0))^2 - i\epsilon \, \text{sgn}(t(\tau) - t(0)) = \frac{4}{a^2} \sinh^2(a\tau/2) - i\epsilon \, \text{sgn}(\tau) \tag{9.33}$$

$$= \frac{4}{a^2} \sinh^2(a\tau/2 - i\epsilon) \tag{9.34}$$

which implies

$$\langle 0| \, \hat{\phi}\big(t(\tau), x(\tau)\big) \, \hat{\phi}\big(t(0), x(0)\big) \, |0\rangle = -\frac{1}{4\pi} \log \bigg(\frac{4}{a^2} \sinh^2(a\tau/2 - i\epsilon) \bigg). \tag{9.35}$$

We now finally compute

$$\frac{d}{d\tau}P_{\text{tot}} = g^2 \int_{-\infty}^{\infty} d\tau \, e^{-iE\tau} \, \langle 0| \, \hat{\phi}\big(t(\tau), x(\tau)\big) \, \hat{\phi}\big(t(0), x(0)\big) \, |0\rangle \tag{9.36}$$

$$= -\frac{g^2}{4\pi} \int_{-\infty}^{\infty} d\tau \, e^{-iE\tau} \log \bigg(\frac{4}{a^2} \sinh^2(a\tau/2 - i\epsilon) \bigg) \tag{9.37}$$

$$= -\frac{g^2}{4\pi} \int_{-\infty}^{\infty} d\tau \frac{1}{iE} e^{-iE\tau} \partial_\tau \log \bigg(\frac{4}{a^2} \sinh^2(a\tau/2 - i\epsilon) \bigg) \tag{9.38}$$

$$= -\frac{g^2}{4\pi i E} \int_{-\infty}^{\infty} d\tau \, e^{-iE\tau} a \frac{\cosh(a\tau/2 - i\epsilon)}{\sinh(a\tau/2 - i\epsilon)} \tag{9.39}$$

$$= -\frac{g^2}{4\pi i E} (-4\pi i) \sum_{n=1}^{\infty} e^{-nE\frac{2\pi}{a}} \tag{9.40}$$

$$= \frac{g^2}{E} \frac{e^{-\frac{2\pi}{a}E}}{1 - e^{-\frac{2\pi}{a}E}}. \tag{9.41}$$

In step (9.38), we integrated by parts and threw out the boundary term (which, while it grows with τ, is also oscillatory, which is why we throw it out). In step (9.40), we closed the contour of integration on the lower half plane due to the dampening factor $e^{-iE\tau}$ and picked up an infinite number of poles, which we draw in Figure 15.

Comparing (9.41) to (9.22), we see that the rate of probability that the Rindler detector goes "bing!" is *exactly* equal to that of a thermal bath of particles with $\beta = 2\pi/a$! Note that this proof didn't use Rindler modes or Rindler quantization or any of that crap—this is just a direct prediction of what happens when we couple an accelerating detector to the vacuum in quantum field theory!!!

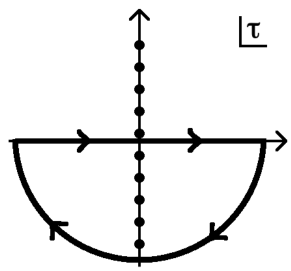

Figure 15: In computing $dP_{\text{tot}}/d\tau$ we close the τ contour on the lower half plane, picking up the poles at $\tau = \frac{2\pi}{a} in + i\epsilon$ for $n = -1, -2, -3, \ldots$.

10 KMS condition

In this section, we will comment on the inherently thermal properties of the two-point function along an accelerating observer's worldline.

Let us first review the properties of two point functions at finite temperature, following [7]. Consider, for instance, the thermal Green's function G_β defined by

$$G_\beta(t, x; 0, 0) \equiv \text{Tr}\Big(e^{-\beta H}\hat{\phi}(t, x)\hat{\phi}(0, 0)\Big). \tag{10.1}$$

Note that this Green's function is translation invariant.

$$G_\beta(t + t', x + x'; t', x') = G_\beta(t, x; 0, 0) \tag{10.2}$$

Using the cyclic property of the trace and $e^{\beta H}\hat{\phi}(t, x)e^{-\beta H} = \hat{\phi}(t - i\beta, x)$, we can show that G_β satisfies the property

$$\begin{aligned} G_\beta(t, x; 0, 0) &= \text{Tr}\Big(e^{-\beta H}\hat{\phi}(t, x)\hat{\phi}(0, 0)\Big) \\ &= \text{Tr}\Big(\hat{\phi}(0, 0)e^{-\beta H}\hat{\phi}(t, x)\Big) \\ &= \text{Tr}\Big(e^{-\beta H}e^{\beta H}\hat{\phi}(0, 0)e^{-\beta H}\hat{\phi}(t, x)\Big) \\ &= \text{Tr}\Big(e^{-\beta H}\hat{\phi}(-i\beta, 0)\hat{\phi}(t, x)\Big) \\ &= G_\beta(-i\beta, 0; t, x) \\ &= G_\beta(-t - i\beta, -x; 0, 0). \end{aligned} \tag{10.3}$$

The equation

$$G_\beta(t, x; 0, 0) = G_\beta(-t - i\beta, -x; 0, 0) \tag{10.4}$$

is called the "KMS condition." If a Green's function satisfies the above equation, it is said to be thermal.

For a Rindler observer on an accelerated trajectory, let us define

$$\begin{aligned} G_+(\tau) &\equiv \langle 0|\, \hat{\phi}\big(t(\tau), x(\tau)\big)\hat{\phi}\big(t(0), x(0)\big)\, |0\rangle \\ &= -\frac{1}{4\pi}\log\left(\frac{4}{a^2}\sinh^2(a\tau/2 - i\epsilon)\right) \end{aligned} \tag{10.5}$$

Notice that

$$G_+(\tau) = G_+(-\tau - i\tfrac{2\pi}{a}). \tag{10.6}$$

Comparing the above equation with (10.4), we see that the vacuum Green's function on an accelerated trajectory has an inverse temperature of $\beta = 2\pi/a$.

Let us now outline an argument from [8] for why equation (10.6) implies that our detector will reach thermal equilibrium over time. Consider two possible scenarios: one where the detector initially starts in the $|\downarrow\rangle$ state and transitions to the $|\uparrow\rangle$ state, and one where the detector starts in the $|\uparrow\rangle$ state and transitions to the $|\downarrow\rangle$ state. Both processes have some probability rate to occur, which we shall denote $\dot{P}_{\downarrow \text{ to } \uparrow}$ and $\dot{P}_{\uparrow \text{ to } \downarrow}$ respectively. Imagine that we keep our detector active for a long time (and it may be decohering during this process) such that both of these processes occur many times. Under a thermal equilibrium, the probability that the detector is in the up state should be $e^{-\beta E}$ times the probability that it is in the down state. This is achieved if

$$\frac{\dot{P}_{\downarrow \text{ to } \uparrow}}{\dot{P}_{\uparrow \text{ to } \downarrow}} = e^{-\beta E}. \tag{10.7}$$

So, is this case? Well, from (9.36), we computed

$$\dot{P}_{\downarrow \text{ to } \uparrow} = g^2 \int_{-\infty}^{\infty} d\tau \, e^{-iE\tau} G_+(\tau) \tag{10.8}$$

and, to get the probability rate to go from $\uparrow$ to $\downarrow$, all we must do is send $E \to -E$ in the above equation to get

$$\dot{P}_{\uparrow \text{ to } \downarrow} = g^2 \int_{-\infty}^{\infty} d\tau \, e^{+iE\tau} G_+(\tau). \tag{10.9}$$

However, using (10.6) on the above equation, we get

$$\begin{aligned} \dot{P}_{\uparrow \text{ to } \downarrow} &= g^2 \int_{-\infty}^{\infty} d\tau \, e^{+iE\tau} G_+(-\tau - i\beta) && (10.10) \\ &= g^2 \int_{-\infty}^{\infty} d\tau' e^{-iE(\tau' + i\beta)} G_+(\tau') && (10.11) \\ &= e^{\beta E} \dot{P}_{\downarrow \text{ to } \uparrow} && (10.12) \end{aligned}$$

which gives exactly (10.7)! Therefore, the KMS condition can be used to prove that the DeWitt detector will reach thermal equilibrium with $\beta = 2\pi/a$.

11 What does an inertial observer see when the Rindler observer detects a particle?

11.1 Answering the question with no math

Let's say you have an accelerating Rindler observer with a particle detector. The detector goes "bing" and, from the Rindler observer's perspective, absorbs a particle. How does an inertial observer explain what happened? What do they think happened when the detector went "bing"? (This was first addressed in [3].)

The basic answer can actually be deduced without using any equations. Because the Rindler observer sees that they have absorbed a Rindler particle from the Minkowski vacuum $|0\rangle$, then they know that the final quantum state is not $|0\rangle$ anymore because a Rindler particle has been subtracted. However, from the inertial observer's perspective, $|0\rangle$ is of course the state with no particles in it.

Crucially, while different observers might describe the same quantum state in different ways, quantum states themselves are *observer independent objects*. That means that if the final quantum state is not $|0\rangle$ from the Rindler observer's perspective, then it cannot be $|0\rangle$ from the inertial observer's perspective either. But if the inertial observer thinks the final quantum state is not $|0\rangle$, then the only possibility is that a particle must have been *produced* (!!!) by the measurement process!

But how can this be? *How* can a particle be created by this detector, according to the inertial observer? Well, the answer is not particularly sophisticated. All measurement devices work by coupling to the object they are trying to measure in some way. This is what allows for the object being measured to affect the measurement device. However, this 'coupling' goes both ways, and the measurement device must always necessarily perturb the object of measurement in some way as well. In fact, looking at the Hamiltonian of the joint detector + quantum field system, we can see that the detector essentially acts as a "source" for the quantum field. So the accelerating detector can radiate particles in a process which is not entirely dissimilar to how an accelerating charge will radiate photons via the process of radiation reaction, or "bremsstrahlung."[5]

11.2 A little more math

So, what is the nature of this particle which has been produced? We will analyze this question in great detail in the next section, but let's first give a heuristic description. When the Rindler observer detects a particle of Rindler energy $\lambda = E/a$ (which can be either right moving or left moving with equal probability), that Rindler particle is removed from the vacuum and the final quantum field state will be roughly

$$|f\rangle \approx |0, \downarrow\rangle + i\varepsilon\big(\hat{a}(\Phi^{L,\lambda}_{\mathrm{I,IV}}) + \hat{a}(\Phi^{R,\lambda}_{\mathrm{I,III}})\big)\,|0, \uparrow\rangle \tag{11.1}$$

where ε is some tiny constant which is unrelated to our $i\epsilon$ proscription from earlier.

Now, the above equation may look a bit strange, because we have annihilation operators acting on the vacuum. However, these annihilation operators actually do *not* annihilate the vacuum, because both $\Phi^{L,\lambda}_{\mathrm{I,IV}}$ and $\Phi^{R,\lambda}_{\mathrm{I,III}}$ are sums of positive and negative inertial frequency functions.

The positive and negative frequency parts of these Rindler modes, are denoted as $(\Phi^{L,\lambda}_{\mathrm{I,IV}})_\pm$

[5]One should not take this analogy too seriously, however, because there are substantial differences between the release of radiation from the DeWitt detector and the release of photons from an accelerating charge.

and $(\Phi^{R,\lambda}_{\text{I,III}})_\pm$, can be deduced from (7.12), (7.13), (7.14), and (7.15) and are

$$(\Phi^{L,\lambda}_{\text{I,IV}})_+ = \frac{e^{\pi\lambda}}{e^{\pi\lambda} - e^{-\pi\lambda}}(v - i\epsilon)^{-i\lambda} \tag{11.2}$$
$$= \frac{1}{e^{\pi\lambda} - e^{-\pi\lambda}}\left(e^{\pi\lambda}\Theta(v)v^{-i\lambda} + \Theta(-v)(-v)^{-i\lambda}\right)$$
$$(\Phi^{R,\lambda}_{\text{I,III}})_+ = \frac{1}{e^{\pi\lambda} - e^{-\pi\lambda}}(u - i\epsilon)^{i\lambda} \tag{11.3}$$
$$= \frac{1}{e^{\pi\lambda} - e^{-\pi\lambda}}\left(\Theta(u)u^{i\lambda} + e^{\pi\lambda}\Theta(-u)(-u)^{i\lambda}\right)$$

$$(\Phi^{L,\lambda}_{\text{I,IV}})_- = \frac{-e^{-\pi\lambda}}{e^{\pi\lambda} - e^{-\pi\lambda}}(v + i\epsilon)^{-i\lambda} \tag{11.4}$$
$$= \frac{-1}{e^{\pi\lambda} - e^{-\pi\lambda}}\left(e^{-\pi\lambda}\Theta(v)v^{-i\lambda} + \Theta(-v)(-v)^{-i\lambda}\right)$$
$$(\Phi^{R,\lambda}_{\text{I,III}})_- = \frac{-1}{e^{\pi\lambda} - e^{-\pi\lambda}}(u + i\epsilon)^{i\lambda} \tag{11.5}$$
$$= \frac{-1}{e^{\pi\lambda} - e^{-\pi\lambda}}\left(\Theta(u)u^{i\lambda} + e^{-\pi\lambda}\Theta(-u)(-u)^{i\lambda}\right).$$

From (6.4), we have that

$$\hat{a}(f)\,|0\rangle = -\hat{a}^\dagger(f^*_-)\,|0\rangle \tag{11.6}$$

and thus, we see that our final state is roughly

$$|f\rangle \approx |0,\downarrow\rangle - i\varepsilon\big(\hat{a}^\dagger((\Phi^{L,\lambda}_{\text{I,IV}})^*_-) + \hat{a}^\dagger((\Phi^{R,\lambda}_{\text{I,III}})^*_-)\big)\,|0,\uparrow\rangle\,. \tag{11.7}$$

This explains how the Rindler observer sees a Rindler particle being absorbed while the inertial observer sees a particle being emitted.

However, if one looks at (11.7) and then looks at wave functions $(\Phi^{L,\lambda}_{\text{I,IV}})_-$ and $(\Phi^{R,\lambda}_{\text{I,III}})_-$ in (11.4), (11.5), one can see that that created particle's wave function is non-zero in all four regions of the spacetime! This defies expectation, because one might naively assume that the created particle could not have any support in region II if the Rindler observer is contained completely within region I. In fact, this *had* to be the case, because positive frequency wave functions cannot vanish on a finite interval of space, as discussed in section 8.

Actually, inspecting the equations (11.4), (11.5) a bit more closely, we can see that the particle's wave function is actually *bigger* in region II than region I by a factor of $e^{\pi\lambda}$!

You might be concerned that we have somehow broken causality. Rest assured, this is not the case. The outcomes of any measurements conducted in region II will not be affected by the emission of this particle. (However, the outcomes of region II measurements *will* be correlated with the outcome of region I measurements, although these correlations cannot be seen until one is in region IV.) This will be discussed in more detail in section 11.6.

11.3 The complete analysis

Let us solve for the exact final state $|f\rangle$ to the first order in g due to a moving detector travelling on an arbitrary trajectory $x(t)$. Afterwards, we will specialize to the case of a stationary worldline and an accelerating worldline and compare the two.

In general, if we have a Hamiltonian $H(t) = H_0 + W(t)$ where H_0 is a free Hamiltonian and $W(t)$ is a tiny perturbation, then we can find the transition matrix $U(t_f; t_i)$ from the quantum state at time t_i to time t_f to first order using Dyson's time ordering symbol $\mathcal{T}$:

$$U(t_f; t_i) = \mathcal{T}\{e^{-i\int_{t_i}^{t_f} dt\,(H_0+W(t))}\} \tag{11.8}$$
$$= \mathcal{T}\{e^{-i\int_{t_i}^{t_f} dt\,H_0}(1 - i\int_{t_i}^{t_f} dt'\,W(t') + \ldots)\} \tag{11.9}$$
$$= e^{-iH_0 t_f}\left(1 - i\int_{t_i}^{t_f} dt\; e^{iH_0 t}W(t)e^{-iH_0 t} + \ldots\right)e^{iH_0 t_i}. \tag{11.10}$$

The terms $e^{-iH_0 t_f}$ and $e^{iH_0 t_i}$ on the last line of the above equation are not particularly interesting, so we will actually evaluate the term in the middle which we shall denote by $\mathcal{S}$.

$$\mathcal{S} \equiv 1 - i\int_{-\infty}^{\infty} dt\; e^{iH_0 t}W(t)e^{-iH_0 t} \tag{11.11}$$

where from section 9.1 we recall that

$$W(t) = g\,\gamma^{-1}\,w(t)\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right)\hat{\phi}(x(t)) \tag{11.12}$$

where $w(t)$ is a window function which we can use to turn the detector on and off. Notice that we then have

$$e^{iH_0 t}W(t)e^{-iH_0 t} = g\,\gamma^{-1}\,w(t)\left(e^{iEt}\,|\uparrow\rangle\langle\downarrow| + e^{-iEt}\,|\downarrow\rangle\langle\uparrow|)\right)\hat{\phi}(t, x(t)). \tag{11.13}$$

We shall now perform a little trick: the above equation has a $\hat{\phi}$ in it, but what we really want is an equation with $\hat{a}$'s and $\hat{a}^\dagger$'s in it. We would therefore like to find a function $f_{t,x}(t', x')$ such that

$$\begin{aligned}\hat{\phi}(t, x) &= \hat{a}^\dagger(f_{t,x}) + \hat{a}(f_{t,x}) \\ &= i\int dx'\big((f^*_{t,x} - f_{t,x})\partial_{t'}\hat{\phi} - \hat{\phi}\partial_{t'}(f^*_{t,x} - f_{t,x})\big)\end{aligned} \tag{11.14}$$

where the above integral is evaluated on the $t' = t$ time slice.

In order for the above equation to hold, the function $f_{t,x}$ must satisfy

$$f_{t,x}(t', x')\Big|_{t'=t} = 0, \qquad \partial_{t'}(f^*_{t,x} - f_{t,x})\Big|_{t'=t} = i\delta(x - x'), \qquad (\partial^2_{t'} - \partial^2_{x'})f_{t,x} = 0. \tag{11.15}$$

Note that the radiative propagator G_{rad} (see appendix F for a review)

$$G_{\text{rad}}(t, x) = \tfrac{1}{4}\text{sgn}(t - x) + \tfrac{1}{4}\text{sgn}(t + x) \tag{11.16}$$

satisfies

$$G_{\text{rad}}(0, x) = 0, \qquad \partial_t G_{\text{rad}}(0, x) = \delta(x), \qquad (\partial_t^2 - \partial_x^2)G_{\text{rad}} = 0. \tag{11.17}$$

Therefore, f is just given by

$$f_{t,x}(t', x') = -\frac{i}{2}G_{\text{rad}}(t' - t, x' - x) \tag{11.18}$$

and we have

$$\hat{\phi}(t,x) = -\frac{1}{2}\hat{a}^\dagger(i\,G_{\text{rad}}(t'-t,x'-x)) - \frac{1}{2}\hat{a}(i\,G_{\text{rad}}(t'-t,x'-x)). \tag{11.19}$$

The above equation is quite useful—it lets us write $\hat{\phi}$ as a sum of creation and annihilation operators.

However, using equation (6.1), we can rewrite the above equation in two more equivalent forms which are a bit cleaner:

$$\hat{\phi}(t,x) = -\hat{a}^\dagger(i\,G_{\text{rad}}(t'-t,x'-x)) \tag{11.20}$$

$$\hat{\phi}(t,x) = -\hat{a}(i\,G_{\text{rad}}(t'-t,x'-x)). \tag{11.21}$$

Plugging the above two equations into (11.13), we can now explicitly write $\mathcal{S}$ as

$$\boxed{\mathcal{S} = 1 + \hat{a}(\psi_f)\,|{\uparrow}\rangle\langle{\downarrow}| - \hat{a}^\dagger(\psi_f)\,|{\downarrow}\rangle\langle{\uparrow}|} \tag{11.22}$$

where the final wave function ψ_f is given by the following integral over the observer's worldline

$$\boxed{\psi_f(t,x) = g\int_{-\infty}^{\infty} d\tau\, w(\tau)e^{-iE\tau}G_{\text{rad}}\big(t-t(\tau),x-x(\tau)\big).} \tag{11.23}$$

The above two equations are the main results of this section. Let us now specialize to the case of the stationary worldline and the accelerating worldline.

11.3.1 Stationary worldline

Let's specialize to the case of a detector travelling on a stationary worldline

$$(t(\tau),x(\tau)) = (\tau,0) \tag{11.24}$$

and for mathematical simplicity take the window function to be $w(\tau)=1$. We then have

$$\psi_f(t,x) = \frac{g}{4}\int_{-\infty}^{\infty} d\tau\, e^{-iE\tau}(\text{sgn}(t-\tau-x)+\text{sgn}(t-\tau+x)) \tag{11.25}$$

$$= \frac{g}{2iE}\big(e^{-iE(t-x)} + e^{-iE(t+x)}\big) \tag{11.26}$$

where we have thrown out the oscillatory boundary terms $e^{\pm i\infty}$ because they average to zero. (See section 12 for a more thorough discussion of these transient boundary terms.)

Looking at equation (11.22), we can therefore see that if the stationary detector is prepared in the $|{\downarrow}\rangle$ state, it can absorb a right or left moving plane wave with energy E, while if it is initially prepared in the $|{\uparrow}\rangle$ state it can emit right or left moving plane waves of energy E.

11.3.2 Accelerating worldline

Now let us analyze the case when the detector moves on an accelerating worldline, with

$$(t(\tau), x(\tau)) = (\frac{1}{a}\sinh(a\tau), \frac{1}{a}\cosh(a\tau)). \tag{11.27}$$

It is convenient to note

$$t(\tau) - x(\tau) = -\frac{1}{a}e^{-a\tau} \tag{11.28}$$

$$t(\tau) + x(\tau) = \frac{1}{a}e^{a\tau}. \tag{11.29}$$

We then have

$$\begin{aligned}
\psi_f(t,x) &= \frac{g}{4}\int d\tau\, e^{-iE\tau}(\text{sgn}(t - x - t(\tau) + x(\tau)) + \text{sgn}(t + x - t(\tau) - x(\tau)) && (11.30)\\
&= \frac{g}{4}\int d\tau\, e^{-iE\tau}(\text{sgn}(t - x + \frac{1}{a}e^{-a\tau}) + \text{sgn}(t + x - \frac{1}{a}e^{a\tau})) && (11.31)\\
&= -\frac{g}{2iE}\Theta(-(t-x))e^{i\frac{E}{a}\ln(-a(t-x))} - \frac{g}{2iE}\Theta(t-x)e^{-i\frac{E}{a}\ln(a(t+x))} && (11.32)\\
&= -g\frac{e^{iE\log(a)/a}}{2iE}\Phi^{R,E/a}_{\text{I,III}} - g\frac{e^{-iE\log(a)/a}}{2iE}\Phi^{L,E/a}_{\text{I,IV}}. && (11.33)
\end{aligned}$$

Looking at equation (11.22), we can see that if the detector is prepared in the $|{\downarrow}\rangle$ state, it can absorb a right or left moving Rindler mode with $\lambda = E/a$, while if it is prepared in the $|{\uparrow}\rangle$ state it can emit such a right or left moving Rindler mode.

11.3.3 Wait, how again can the accelerating detector *emit* a particle?

There is an apparent paradox I would like to discuss, because I think its resolution is interesting. Imagine we start in the vacuum state and have a DeWitt detector travelling on an arbitrary trajectory, prepared in the $|{\downarrow}\rangle$ state. If we are interested in analyzing the amplitude that it emits a particle with positive frequency wave function ψ, i.e. with

$$|i\rangle = |0,\downarrow\rangle, \qquad |f\rangle = \hat{a}^\dagger(\psi)\,|0,\uparrow\rangle \tag{11.34}$$

then from Fermi's golden rule (using an analysis similar to the one of section 9.2) the transition amplitude $c_{i\to f}$ for this event is

$$c_{i\to f} = -ig\int d\tau\; w(\tau)e^{iE\tau}\psi^*\big(t(\tau), x(\tau)\big) \tag{11.35}$$

If we assume that the detector is "on" for a long time (with $w(t) = 1$) then the above integral will only be non-negligibly large for ψ's such that $\psi(t(\tau), x(\tau)) \sim e^{iE\tau}$. For an inertial observer, there will clearly be no such ψ's which are positive inertial frequency modes, because the phase $e^{iE\tau}$ with $E > 0$ has a negative frequency!

In fact, we appear to have an apparent paradox on our hands. How can we *ever* have a positive frequency wave function ψ which oscillates as $e^{iE\tau}$ on a worldline?

The resolution lies in the fact that the modes $(v - i\epsilon)^{-i\lambda}$ and $(u - i\epsilon)^{i\lambda}$ are positive frequency with respect to inertial time, yet *negative* frequency with respect to Rindler time in region I! (See sections 7.2 and 8 for a discussion of this phenomenon.) In other words, while these modes really are comprised of positive frequency plane waves, the phase of these wave functions when restricted to worldlines in region I has a negative frequency! So while these are positive frequency modes, their "apparent" frequency is negative in half of the spacetime.

11.4 Isn't energy conservation violated?

There is something disconcerting about all of this. If an accelerating DeWitt detector starts in the $|\downarrow\rangle$ state, there is some probability that it will transition into the $|\uparrow\rangle$ state, increasing its rest energy (and rest mass) by E, while also emitting a Minkowski particle, which also increases the energy of the spacetime. Doesn't this break energy conservation?

Obviously it can't. But why not?

Firstly, I will point out that if one looks at the full Hamiltonian $H(t)$ of the system, then the DeWitt detector acts like a source for the quantum field. If this source turns on/off or moves around in the spacetime, then our Hamiltonian $H(t)$ won't be time independent, and by the converse of Noether's theorem, energy will not be conserved. So in that sense energy really isn't conserved in our simplified scenario.

However, obviously our idealized scenario is not taking certain things into account. Most notably, *what is accelerating the detector to begin with*? In reality, there would have to be some energy source which is accelerating it! Somehow, this extra energy source must resolve the paradox of the apparent increase in energy. Having said that, to my knowledge there has not been a full analysis which accounts for the complete balance of energy conservation in Unruh emission. One important effect to keep in mind is that when the particle is emitted, there will be some recoil due to conservation of momentum which will change the trajectory of the detector. The external agent accelerating the detector may then have to expend addition energy to get the detector back on its old trajectory. (See [9] for a recent analysis of momentum recoil due to emission.) It is also possible that a resolution to this question may require analyzing the problem to higher orders in g and/or taking into account the small interaction energy $W = g(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|)\hat{\phi}(x(t))$ when calculating the energy balance of the process. In my opinion, it's an embarrassment to the field that no one has conducted a definitive energy conservation analysis, although I just tried for myself and found it to be quite tricky and confusing.

11.5 Can you extract infinite energy from the vacuum?

Say you travel on an accelerated trajectory with your particle detector. There is some probability that it will go "bing!", transition to its higher energy state and absorb a Rindler particle from the vacuum. What if you to then empty the newfound energy of your detector into some other energy reservoir and repeat the process over again. Could you extract an infinite amount of energy from the vacuum on your accelerated trajectory?

To answer this question, we need to look carefully at what happens after you absorb the first Rindler particle. Crucially, the probability of absorbing a particle is larger the more particles there are. Initially, while the probability of there being n particles in some mode is proportional to $e^{-n\beta E}$, the probability of *detection* is proportional to $ne^{-n\beta E}$. This is because the detector essentially acts with the annihilation operator on the state, and the annihilation operator acts as $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$. Therefore, after your first measurement, a Rindler quanta is subtracted from the vacuum but the expected Rindler energy of the state actually *increases*, because it is more likely that the state you absorbed the quanta from had more particles in it to begin with. In other words, the relative probability that there were n particles in that mode went from $e^{-\beta nE}$ to $ne^{-\beta(n-1)E}$.

However, this wont work forever. Each time you perform a measurement you are taking Rindler quanta out of the state, and eventually the probability of detecting one more quanta will become very low. So no, you cannot extract an infinite amount of energy out of the vacuum in this way, as eventually the Rindler particles will "run out."

11.6 Causality

11.6.1 Density matrix with particle emission

In the last section, we showed that from an inertial perspective, the detection of radiation from an accelerated traveller in region I emits a particle whose wave function has support in both regions I and II. In fact, the wave function's amplitude is actually a factor of $e^{E/a}$ *bigger* in region II than region I! It is therefore warranted for us to ask how causality can be preserved in this scenario.

"Causality" in this context means that the outcomes of measurements conducted in region II cannot depend on whether or not there is a Rindler experimenter in region I who is detecting/emitting particles.

The outcomes of all measurements in region II can be determined by the density matrix obtained by tracing out region I. The vacuum density matrix in region II is denoted by

$$\rho_{\text{vac},\mathrm{II}} = \mathrm{Tr}_{\mathrm{I}}(|0\rangle\langle 0|). \tag{11.36}$$

Meanwhile, if there is an accelerating Rindler detector in region I which emitted a particle, then the final state is

$$|\Psi\rangle = |0,\downarrow\rangle + \frac{g}{2iE}\left(\hat{a}(\Phi^{L,E/a}_{\mathrm{I,IV}}) + \hat{a}(\Phi^{R,E/a}_{\mathrm{I,III}})\right)|0,\uparrow\rangle \tag{11.37}$$

up to some phases. We denote the region II density matrix of the above state, found by tracing out region I and the detector's two-level hilbert space $\mathcal{H}_{\text{d}} = \{|\uparrow\rangle, |\downarrow\rangle\}$, as

$$\rho_{\Psi,\mathrm{II}} = \mathrm{Tr}_{\mathrm{I,d}}(|\Psi\rangle\langle\Psi|). \tag{11.38}$$

Using the expression for the vacuum state (7.24), it is trivial to verify that the density matrix of region II for the above state is equal to the density matrix of region II in the vacuum:

$$\rho_{\text{vac},\mathrm{II}} = \rho_{\Psi,\mathrm{II}}. \tag{11.39}$$

Note that we are only justified in computing the density matrix $\rho_{\Psi,\mathrm{II}}$ to linear order in g, and the only $\mathcal{O}(g)$ terms which appear in the trace computation are of the form $g\ {}_L\langle n,\mathrm{I}|n-1,\mathrm{II}\rangle_L$ and $g\ {}_R\langle n,\mathrm{I}|n-1,\mathrm{II}\rangle_R$, both of which vanish. If we wanted to compute the density matrix to higher orders in g, we would have to compute the Dyson series to higher orders in g, although if we were to do so we would still find that $\rho_{\text{vac},\mathrm{II}} = \rho_{\Psi,\mathrm{II}}$. Because the density matrix in region II is unaffected by the presence of the DeWitt detector moving along the Rindler worldine in region I, we can conclude that causality is respected.

11.6.2 The example of the $u = 0$ pulse

But you may still be bothered. How can causality be respected if the particle's wavefunction is present in both regions I and II, and even larger in region II? It seems easy to concoct blatant contradictions. For instance, this particle state will have a non-zero stress energy energy in region II! So, what is the resolution?

It is crucial to remember that the Rindler detector only has a certain probability to emit a particle. It does not emit a particle 100% of the time. So we need to take into account that our state is a superposition of basis states with *different particle number*. The fact that we have a precise superposition of states of different particle number is how measurements in region II can unaffected by the presence of such particles, even if the particles themselves have wave functions which are supported in region II.

Let's see how this works in a simplified context. Imagine a particle state that is the positive frequency part of a Dirac delta "pulse" at $u = t - x = 0$. Using the decomposition of the Dirac delta function $\delta(\alpha)$

$$\delta(\alpha) = \frac{1}{2\pi i}\left(\frac{1}{\alpha - i\epsilon} - \frac{1}{\alpha + i\epsilon}\right) \tag{11.40}$$

we may denote a $u = 0$ "pulse" function as $\delta_u(t, x)$, where

$$\delta_u(t, x) \equiv \delta(t - x), \tag{11.41}$$

then we can break δ_u up into positive and negative frequency parts

$$\delta_u = \delta_{u,+} + \delta_{u,-} \tag{11.42}$$

where

$$\delta_{u,+}(t, x) = \frac{1}{2\pi i}\frac{1}{t - x - i\epsilon}, \qquad \delta_{u,-}(t, x) = -\frac{1}{2\pi i}\frac{1}{t - x + i\epsilon}. \tag{11.43}$$

For convenience, we remind the reader of the the commutators (5.19) and (5.24), copied below as

$$[\hat{\phi}(t, x), \hat{a}^\dagger(f)] = f(t, x), \tag{11.44}$$

$$[\hat{a}(f), \hat{\phi}(t, x)] = f^*(t, x). \tag{11.45}$$

Using the above commutators, we can see that the state $\hat{a}^\dagger(\delta_{u,+})\,|0\rangle$ has the single particle wave function

$$\langle 0|\,\hat{\phi}(t, x)\hat{a}^\dagger(\delta_{u,+})\,|0\rangle = \delta_{u,+}(t, x) = \frac{1}{2\pi i}\frac{1}{t - x - i\epsilon} \tag{11.46}$$

which is clearly non-zero everywhere in the space time.

Let us now consider the quantum field state $|\varphi\rangle$, defined as

$$|\varphi\rangle = |0\rangle + \varepsilon\, a^\dagger(\delta_{u,+})\,|0\rangle \tag{11.47}$$

where ε is a tiny real constant (unrelated to our $i\epsilon$ proscription) which is analogous to our tiny coupling g. Imagine measuring the field value at a point (t, x) with the field operator $\hat{\phi}(t, x)$.

The expectation value would be

$$
\begin{aligned}
\langle\varphi|\,\hat{\phi}(t,x)\,|\varphi\rangle &= \langle 0|\,\big(1+\varepsilon\,\hat{a}(\delta_{u,+})\big)\hat{\phi}(t,x)\big(1+\varepsilon\,\hat{a}^\dagger(\delta_{u,+})\big)\,|0\rangle \\
&= \varepsilon\,\delta_{u,+}(t,x)+\varepsilon\,\delta^*_{u,+}(t,x)+\langle 0|\,\hat{\phi}(t,x)\,|0\rangle \\
&= \varepsilon\,\delta(t-x)+\langle 0|\,\hat{\phi}(t,x)\,|0\rangle\,.
\end{aligned}
\tag{11.48}
$$

So, while the particle with wave function $\delta_{u,+}$ can be "present" anywhere in the spacetime, in the state $|\varphi\rangle$ its presence will only affect measurements on the line $t-x=0$! This state $|\varphi\rangle$ is therefore analogous to the state $|\Psi\rangle$ from (11.37) we encountered earlier in our study of particle emission from the Unruh effect, as in the state $|\Psi\rangle$ we showed that measurements conducted in region II will be indistinguishable from measurements conducted in the vacuum state.

What if we were to make ε finite and take higher orders of ε into account? Let us define the unitary operator U by

$$
U \equiv \exp\big(\varepsilon\,\hat{a}^\dagger(\delta_{u,+})-\varepsilon\,\hat{a}(\delta_{u,+})\big). \tag{11.49}
$$

Using the commutator

$$
\begin{aligned}
[\hat{\phi}(t,x),\hat{a}^\dagger(\delta_{u,+})-\hat{a}(\delta_{u,+}))] &= \delta_{+,u}+\delta_{-,u} \\
&= \delta(t-x)
\end{aligned}
\tag{11.50}
$$

we see that $\hat{\phi}(t,x)$ commutes with U if $t-x\neq 0$.

$$
[\hat{\phi}(t,x),U]=0 \quad \text{if } t-x\neq 0. \tag{11.51}
$$

Therefore, the expectation value of $\hat{\phi}(t,x)$ in the state $U\,|0\rangle$ will be equal to its expectation value in the state $|0\rangle$, assuming $t-x\neq 0$:

$$
\begin{aligned}
\langle 0|\,U^\dagger\hat{\phi}(t,x)U\,|0\rangle &= \langle 0|\,U^\dagger U\hat{\phi}(t,x)\,|0\rangle \\
&= \langle 0|\,\hat{\phi}(t,x)\,|0\rangle\,.
\end{aligned}
\tag{11.52}
$$

Because we may describe the state $U\,|0\rangle$ as being a specific superposition of the multiparticle states $(\hat{a}^\dagger(\delta_{+,u}))^n\,|0\rangle$, we can conclude from the above above analysis that these kind of "detailed superposition" states can have particles "present" all throughout the spacetime while leaving measurements in certain regions of the spacetime unaffected, such as the region $t-x\neq 0$ in the above example.

There are a few key things here worth pointing out. Any sort of physical process that you could carry out in the real world can only transform states via *unitary* operators. Such operators can be expressed as the exponentials of *skew-adjoint* operators. Notice that creation operators $\hat{a}^\dagger$ are not skew-ajoint, while the combination $\hat{a}^\dagger-\hat{a}$ is skew adjoint.

Therefore, we can't create a unitary operator by exponentiating the particle creation operator $\hat{a}^\dagger(\delta_{u,+})$— we instead need to exponentiate the skew-adjoint combination of the creation and annihilation operator $\hat{a}^\dagger(\delta_{u,+})+\hat{a}(\delta_{u,+})$ as we did in (11.50). In fact, there is no unitary operator that only ever *creates* a particle, i.e. there is no unitary U for which $U\,|n\rangle=|n+1\rangle$. Such an operator U can never be unitary simply because it is non-invertable, as no state is sent to the state with occupation number $n=0$. This implies that all physical, unitary processes must posses the possibility to both *create and annihilate* particles, which will end up being balanced in detailed superpositions in such a way as to be undetectible outside of a lightcone of influence.

11.6.3 Lightcone of influence of a source

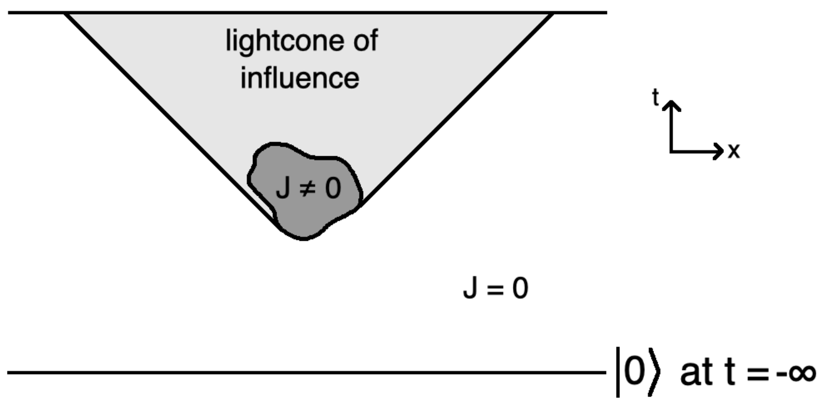

Figure 16: If a source is $J(t,x)$ is non-zero in some region of spacetime and if the quantum field state starts at $|0\rangle$ at $t = -\infty$, then only measurements conducted in the future lightcone of the region where $J(t,x) \neq 0$ will be different from measurements conducted in the vacuum.

Let us elaborate more on how QFT measurements will be unaffected by perturbations outside of a "lightcone of influence" by studying the following model.

Say we have a background source $J(t,x)$ which couples to our scalar field as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + J\phi. \tag{11.53}$$

The time dependent Hamiltonian of our scalar field is then

$$H(t) = H_0^{\text{QFT}} - \int dx\, J(t,x)\hat{\phi}(x). \tag{11.54}$$

In the Heisenberg picture, operators evolve as

$$\frac{d}{dt}\mathcal{O}(t) = i[H(t), \mathcal{O}(t)]. \tag{11.55}$$

In particular, we have

$$\frac{d}{dt}\hat{\phi}(t,x) = \hat{\pi}(t,x) \tag{11.56}$$

$$\frac{d}{dt}\hat{\pi}(t,x) = \partial_x^2 \hat{\phi}(t,x) + J(t,x) \tag{11.57}$$

and can combine the above two equations as

$$(\partial_t^2 - \partial_x^2)\hat{\phi}(t,x) = J(t,x). \tag{11.58}$$

We can use a Green's function to solve this equation. Following appendix F, we define $G_{\text{ret}}(t,x)$ to the solution to the inhomogeneous wave equation

$$(\partial_t^2 - \partial_x^2)G_{\text{ret}}(t,x) = \delta(t)\delta(x) \tag{11.59}$$

with the boundary condition that is vanishes in the past, i.e.

$$G_{\text{ret}}(t, x) = 0 \text{ for } t < 0. \tag{11.60}$$

In particular, $G_{\text{ret}}(t, x)$ is only non-zero inside of the future lightcone of the origin.

Let us now assume that our quantum field state starts off in the vacuum $|0\rangle$ at $t = -\infty$, and at some finite time the source J is turned on.

Because we specified that our state started in the vacuum in the past, we need to use the retarded Green's function (as opposed to its counterpart, the time-reversed advanced Green's function) to write a general solution of $\hat{\phi}(t, x)$ in the Heisenberg picture as

$$\hat{\phi}(t, x) = \hat{\phi}(t, x)_{J=0} + \int dt' dx' \, J(t', x') G_{\text{ret}}(t - t', x - x') \tag{11.61}$$

where $\hat{\phi}_{J=0}$ solves the free equation $(\partial_t^2 - \partial_x^2)\hat{\phi}_{J=0} = 0$ and is what $\hat{\phi}$ *would have been* if the source was not present.

From equation (11.61), we can see that the presence of an external background source J will only affect quantum field measurements inside of the lightcone of of influence of J. In the Heisenberg picture, the DeWitt detector itself can be thought of as a source travelling on an accelerated trajectory, and therefore will only affect the outcome of quantum field measurements in regions I and IV.

12 Transient effects (and an apologia)

In this note, we have generally taken the detection time T to be very large. However, if we were to instead take an inertial detector and quickly turn it on and off, we would find that even in this case there would be some probability for the detector to go "bing!" and emit a particle. We do not, however, take this as evidence that the Minkowski vacuum is thermal as seen by inertial observers. We just attribute this behavior the the turning on-and-off of the detector itself. Indeed, once the detector is turned on, the Minkowski vacuum is no longer in a definite energy state and thus there must be some amplitude for particles to be emitted. This is remedied by keeping the detector on for a long time T.

However, there is a separate class of transient effects we have encountered in this note which are slightly more dire. At multiple points in this note, we have computed integrals by flagrantly throwing out oscillating boundary terms.[6] Sometimes I noted this in the text, other times I did not. I feel a bit guilty about this, so I will give you a complete list of times I did this here: the integration of parts between (9.37) and (9.38), and throwing out the $e^{\pm i\infty}$ boundary terms in going from (11.25) to (11.26) as well as (11.31) to (11.32).

Throwing out these oscillatory boundary terms can be justified using the following consideration. In this note, we have exclusively calcuated probability rates of transition using first order perturbation theory. As we explicitly spelled out in section 9.2, while T is "large," it is also "small" in the sense that $gT \ll 1$ and our transition amplitudes are much less 1. However, if we imagine that we are keeping our detector active for a *very* long time, during which many such detection events can occur and the detector can decohere during this process, then the exact length of T

[6] I'm not counting the times I used the $i\epsilon$ regulator, which is completely legit.

will vary randomly from one detection to the next. The variable length of the detection time T means that oscillatory terms in T will "average to zero," and thus we are justified in throwing them out.

However, there are *still* some mathematical steps I undertook which are even more problematic, and have to do with the particular DeWitt detector I used for a massless field in 1+1 dimensions.

The problem essentially has to do with the fact that the non time-ordered two-point function for a massless field in 1+1d is logarithmic, with

$$\langle 0| \hat{\phi}(t,x)\hat{\phi}(0,0)|0\rangle = -\frac{1}{4\pi}\log\big(t^2 - x^2 - i\epsilon\,\mathrm{sgn}(t)\big). \tag{12.1}$$

In higher dimensions, the two-point function is a power law and goes to zero for far-separated points. Meanwhile, in 1+1d, the two point function actually grows for far-away points.

In particular, on an accelerated trajectory the two point function is

$$\begin{aligned} G_+(\tau) &\equiv \langle 0| \hat{\phi}\big(t(\tau), x(\tau)\big)\hat{\phi}\big(t(0), x(0)\big)|0\rangle \\ &= -\frac{1}{4\pi}\log\left(\frac{4}{a^2}\sinh^2(a\tau/2 - i\epsilon)\right) \end{aligned} \tag{12.2}$$

which actually grows *linearly* for large τ!!! Therefore, when we integrated by parts to go from (9.37) to (9.38), the boundary term we threw out was no only oscillatory but also growing[7] with τ as $\tau \to \infty$. What possible justification could we have for *that*!? Furthermore, the linear growth of the two point function actually impeded another mathematical step we undertook. When we silently deforming the contour of τ integration from $\pm\infty$ to $\pm\infty + i\beta$ to go from equation (10.10) to equation (10.11), this deformation was technically not allowed because $G_+(\tau)$ does not die off at $\tau = \pm\infty$.

So, what is going on here, and what is my excuse for lying? This issue is specific to the exact kind of detector coupling we are using for our 1+1d massless field. The aforementioned growing oscillations are due to the fact that there are many modes which are arbitrarily close to the detector energy E. In other words, as we noted in (9.18), in order to replace the oscillatory integral by a Dirac delta function we need $(E - |k|)T \gg 1$, however there is always *some* $|k|$ close enough to E to run afoul of this no matter how large T is. The problems with the two-point function are really manifestations of this problem, which goes away in higher dimensions.

However, there is actually a way to fix this problem even in 1+1d. In the interaction term in the Hamiltonian $W(t)$ from (9.4), all we must do is replace $\hat{\phi}(x(\tau))$ with $\partial_\tau\hat{\phi}(x(\tau))$. This has the effect of differentiating $G_+(\tau)$ by ∂_τ twice, changing the frightening logarithm into a friendly inverse power law. (The time derivatives have the effect of dampening the effects from field modes arbitrarily close to the detector energy E.) The reason I did not use this modified detector coupling $\partial_\tau\hat{\phi}(x(\tau))$ is because, while better behaved, I wanted to strive for conceptual clarity in this note, and many equations would become much "messier looking" with this change. However, I will use the fact that I *could have* done this if I wanted to as justification for not doing so.

[7] I will mention that throwing out a boundary term which is both oscillating and growing is actually not so crazy, as a similar phenomenon occurs with the derivative of the Dirac function if one looks at the Fourier transorm $\delta'(\alpha) = \frac{i}{2\pi}\int dk\; k e^{ik\alpha}$, as $\delta'(\alpha) = 0$ for essentially all α.

13 Unruh's "particle in a box" detector

In this note, we have used a very simple two-level system for our model particle detector. But what about more realistic particle detectors?

Let's briefly discuss Unruh's original "particle in a box" particle detector from [2, 10]. In these papers, Unruh considers a non-relativistic particle in a box in 3+1 dimensions which satisfies a Schrodinger-like equation. The detector particle is coupled to the scalar field (with coupling constant g) akin to how an electron is coupled to the electromagnetic field. For simplicity, we say the size of the box is much smaller than the acceleration length scale a^{-1}.

When we put the box on an accelerated trajectory, the non-relativistic particle feels a linear potential in the box akin to that of a gravitational field, deforming the wave functions slightly. The particle's eigenfunctions will evolve as

$$e^{-iE_n\tau} h_n(x) \tag{13.1}$$

where $h_n(x)$ are the orthonormal energy eigenwavefunctions and τ is the proper time experienced by the box.

Let's say we begin with the box in the ground state $h_0(x)$. If we accelerate the box and start the quantum field in the Minkowski vacuum, then the transition probability rate for the box to transition from the ground state to the n^{th} excited state is

$$\frac{d}{dT} P_{0\to n} = \lim_{T\to\infty} \frac{g^2}{T} \sum_i \left| \int_{\text{box}} d^3x \int_0^T d\tau \; e^{i(E_n - E_0)\tau} \Phi_i^*(\tau, x) h_0(x) h_n^*(x) \right|^2 \tag{13.2}$$

where $\Phi_i(\tau, x)$ are the positive inertial frequency modes with i labelling an orthonormal set of such modes. Because $E_n - E_0 > 0$, we can see that the transition probability will only be non-zero if there is a positive frequency mode which goes as $\Phi_i(\tau, x) \sim e^{i\omega\tau}$ with $\omega > 0$ along the Box's trajectory. Of course, from our discussion in section 11.3.3, we know that there do exist such modes and the transition probability is therefore non-zero.

What we can see from the above formula is that the transition rate for this more realistic detector is essentially the same as for our simple two-level DeWitt detector. The key property is that both detectors couple to the quantum field using their own proper time, not inertial time.

I will point out one key difference that exists between our two-level DeWitt detector and the particle in a box. To first order in perturbation theory, the two-level DeWitt detector had no amplitude for it's internal state to stay the same from a transition, i.e. for $|\uparrow\rangle \to |\uparrow\rangle$ or $|\downarrow\rangle \to |\downarrow\rangle$. This is because the interaction term in the Hamiltonian was of the form $|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|$. This is *not* the case for the particle in the box, which can undergo "transitions" from $E_n \to E_n$. While this fact in some ways makes its analysis a bit more annoying, it does not provide a qualitative change to our previous discussions.

We are now in a position to answer some common questions about realistic Unruh radiation detection. For instance, would an actual accelerated observer see the particles coming from all directions isotropically, like a stationary observer in a thermal bath of particles would? Let us first quote Bill & Bob themselves on this question:

> We emphasize that Eq. (7.26) is *exactly* a thermal density matrix for Rindler states. Any particle detector used by the accelerating observer which measures the state of

the field in terms of Rindler particles will determine that there is a thermal distribution of Rindler particles. However, this does *not* mean that a detector will respond in exactly the same way as it would if placed in inertial motion in a (real) thermal bath of Minkowski particles. This is because the mode functions $\Phi^{L/R,\lambda}_{\mathrm{I/II}}$ for Rindler particles are different from the mode functions for Minkowski particles. Another way of saying this is that the properties of a box of thermal radiation in inertial motion are measurably different from those of a box of thermal radiation in accelerating motion;[11] for example the density distribution in the inertial box is uniform, whereas the density distribution in the accelerating box will vary with height because of the effective gravitational field. In the case of a scalar field, it turns out that a "monopole detector" cannot distinguish between the inertial and accelerating thermal distributions. However, in the case of an electromagnetic field, this difference can be seen [12, 13]. We emphasize, however, that an accelerating observer still sees an *exactly* thermal distribution of particles; the only difference between the scalar and electromagnetic cases is that in the electromagnetic case, and isotropic point detector is sensitive to the fact that the mode functions in the accelerating case are different from the mode functions in the inertial case.

—Bill Unruh and Bob Wald in [3], 1983

Bill & Bob make the key point that while the region I density matrix is exactly thermal with respect to the boost Hamiltonian, the Rindler modes have different wave functions from the inertial modes. Most notably, while inertial plane waves behave as e^{ikx} on a $t = 0$ slice, the Rindler modes behave as $x^{i\lambda} = e^{i\lambda \log(x)}$. This causes some differences in detection between an inertial detector in a thermal bath of particles and an accelerating detector in the vacuum, although we did not see such a difference for our monopole DeWitt detector. Notably, in [13], the authors studied a single electron detector with an intrinsic two-level magnetic dipole moment. They found that the exact transition rate between the up and down spin states was *not* the same for an accelerating electron as for an inertial electron bathed in a thermal bath of photons. Nonetheless, once *equilibrium* is reached, the accelerating electron's spin degree of freedom will still be found to be in an exact thermal superposition. This can seen from the proof that thermal equilibrium is reached using the KMS condition we gave in section 10. That proof wasn't concerned with the exact transition probabilities, but only the *ratio* of the transition probabilities between going from $\uparrow$ to $\downarrow$ and $\downarrow$ to $\uparrow$. The fact that both accelerated transition probabilities may not be exactly equal to their inertial counterparts doesn't matter, as long as the transition probabilities are only affected up to an overall multiplicative constant. In fact, the proof we gave in 10 really only depended on the fact that G_+ was a function of the quantity

$$(t(\tau_1) - t(\tau_2))^2 - (x(\tau_1) - x(\tau_2))^2 = (4/a^2)\sinh^2(a(\tau_1 - \tau_2)/2) \tag{13.3}$$

on the worldline. This will always be trivially satisfied via Lorentz invariance, which proves that thermal equilibrium is guaranteed to be reached for any kind of pointlike detector, whether the interaction term behaves as a monopole, dipole, quadrupole, etc.

Let us return to the question of whether the radiation is detected isotropically. I am pretty sure the issue can be illuminated with the following consideration. Say one is accelerating in a spaceship of finite length. If the proper length of the spaceship stays constant, then the proper acceleration at the tail of the spaceship will be greater than the proper acceleration at the tip of the spaceship, and therefore the Unruh temperature at the tail is greater than the Unruh temperature at the tip. A pointlike dipole is like the zero-size limit of this situation, as it is a superposition of two monopoles with a tiny separation. Therefore, any detection method which depends on the finite extent of the detector will "feel" the Unruh radiation differently than an inertial bath of radiation, while a true point will not. Unfortunately, I must admit that I don't feel particularly qualified to give any exact details on the non-isotropic nature of such detection, as it is quite a hairy thing to analyze thoroughly.

14 The uncertainty principle and wave functionals

14.1 Proof of Unruh effect from the uncertainty principle

In this section, I would like to give a qualitatively different explanation for the Unruh effect, arguing that it is essentially a manifestation of the uncertainty principal.

Let us briefly restore $\hbar$ and discuss the 1d harmonic oscillator. From the uncertainty principle, the lowest energy state of the harmonic oscillator has a non-zero expectation value for $\langle q^2 \rangle$ and $\langle p^2 \rangle$. This is a quantum effect, because classically you would have $\langle q^2 \rangle = \langle p^2 \rangle = 0$. Using

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\Big(\hat{q} + \frac{i}{m\omega}\hat{p}\Big), \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\Big(\hat{q} - \frac{i}{m\omega}\hat{p}\Big). \tag{14.1}$$

$$\hat{q} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}), \qquad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}^\dagger - \hat{a}) \tag{14.2}$$

we can calculate

$$\left\langle q^2 \right\rangle = \langle 0|\, \hat{q}^2 \,|0\rangle = \frac{\hbar}{2m\omega}. \tag{14.3}$$

Notice that as $\hbar \to 0$, the above expectation value vanishes. If one were to put the 1d quantum oscillator at a finite temperature, with

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-n\beta\hbar\omega} = \frac{1}{1 - e^{-\beta\hbar\omega}}. \tag{14.4}$$

then the $\langle q^2 \rangle$ expectation value become

$$\left\langle q^2 \right\rangle_\beta = \sum_{n=0}^{\infty} \frac{e^{-n\beta\hbar\omega}}{Z(\beta)} \langle n|\, \hat{q}^2 \,|n\rangle = \frac{\hbar}{2m\omega} \coth(\beta\hbar\omega/2) \tag{14.5}$$

We can therefore see that the $\coth(\beta\hbar\omega/2)$ factor is the signature of thermality. If we fix β and take $\hbar \to 0$, then $\langle q^2 \rangle \to (m\omega^2\beta)^{-1}$ and is finite.

In quantum field theory, the analog of $\hat{q}$ and $\hat{p}$ are $\hat{\phi}_k$ and $\hat{\pi}_k$, which are the Fourier transforms of the field and momentum operators. On the $t = 0$ slice, we have

$$\hat{\phi}(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \hat{\phi}_k e^{ikx}. \tag{14.6}$$

By the way, because $\hat{\phi}(x)^\dagger = \hat{\phi}(x)$, we have

$$\hat{\phi}_k^\dagger = \hat{\phi}_{-k} \tag{14.7}$$

so these $\hat{\phi}_k$ operators are not self adjoint. We could of course decide to work with the self adjoint combinations such as $\hat{\phi}_k + \hat{\phi}_{-k}$ and $i(\hat{\phi}_k - \hat{\phi}_{-k})$, but we shall not. If we normalize the creation and annihilation operators as

$$[\hat{a}_{k_1}, \hat{a}_{k_2}^\dagger] = 4\pi |k_1| \delta(k_1 - k_2) \tag{14.8}$$

then from the $t = 0$ mode decomposition

$$\hat{\phi}(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{2|k|} (\hat{a}_k e^{ikx} + \hat{a}_k^\dagger e^{-ikx}) \tag{14.9}$$

we have

$$\hat{\phi}_k = \frac{1}{\sqrt{2\pi}} \frac{1}{2|k|} (\hat{a}_{-k}^\dagger + \hat{a}_k) \tag{14.10}$$

and can easily calculate

$$\langle \phi_{k_1} \phi_{k_2} \rangle = \langle 0 | \hat{\phi}_{k_1} \hat{\phi}_{k_2} | 0 \rangle = \frac{1}{2|k_1|} \delta(k_1 + k_2). \tag{14.11}$$

The above equations are just the QFT generalization of the harmonic oscillator ground state q^2 expectation value (14.3). In other words, the above equations characterize the size of the "spread" of expectation values of ϕ_k in the ground state Gaussian wave functional. We shall therefore refer to (14.11) as our key equation which results from the uncertainty principle, and use it to derive the Unruh effect.

It should be noted that while we use $\langle \ldots \rangle$ to denote the quantum expectation value of operators in the ground state, it is also possible to imagine such expectation values arising in more general contexts. For instance, perhaps ϕ is a classical field which is "noisy" for some reason or other, with such classical noise characterized by expectation values (14.11). This was exactly the perspective taken by [12], which also noted that the requirement of a Lorentz invariant spectrum fixes the value of $\langle \phi_{k_1} \phi_{-k_2} \rangle$ up to an overall multiplicative constant.

If one were to find a classical scenario with the right spectrum of noise, we would have an exact classical analog to the Unruh effect. This was done in [14], where a classical analog of the Unruh effect was experimentally measured using water waves.

In a certain sense, the Unruh effect has more to do with the correlation of noise than anything else. All that fussing about with positive and negative frequency modes is secretly just a manifestation of this.

Anyway, enough talk—let us now go about deriving the Unruh effect from the uncertainty principle. We want to show that the ground state $|0\rangle$ gives thermal expectation values with respect to modes which have constant frequency to an accelerated observer. These modes are, of course, the Rindler modes, which on a $t = 0$ slice take the form $\Theta(x) e^{i\lambda \log(x)}$ and $\Theta(-x) e^{i\lambda \log(-x)}$. Let us therefore use the non-standard mode decomposition on the $t = 0$ slice

$$\hat{\phi}(x) = \int_{-\infty}^{\infty} \frac{d\lambda}{\sqrt{2\pi}} (\hat{\phi}_{\lambda,\mathrm{I}} \Theta(x) x^{i\lambda} + \hat{\phi}_{\lambda,\mathrm{II}} \Theta(-x)(-x)^{i\lambda}). \tag{14.12}$$

The operators $\hat{\phi}_{\lambda,\mathrm{I}}$ and $\hat{\phi}_{\lambda,\mathrm{II}}$ are an accelerating observer's analog to $\hat{\phi}_k$, and satisfy

$$\hat{\phi}^\dagger_{\lambda,\mathrm{I}} = \hat{\phi}_{-\lambda,\mathrm{I}}, \qquad \hat{\phi}^\dagger_{\lambda,\mathrm{II}} = \hat{\phi}_{-\lambda,\mathrm{II}}. \tag{14.13}$$

We shall now give the expression for $\hat{\phi}_{\lambda,\mathrm{I}}$ in terms of $\hat{\phi}_k$. Using the following resolution of the Dirac delta function

$$\frac{1}{2\pi}\int_0^\infty \frac{dx}{x} x^{i(\lambda_1-\lambda_2)} = \delta(\lambda_1-\lambda_2) \tag{14.14}$$

we have

$$\hat{\phi}_{\lambda,\mathrm{I}} = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{dx}{x} x^{-i\lambda}\hat{\phi}(x) \tag{14.15}$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{dx}{x} x^{-i\lambda}\int_0^\infty \frac{dk}{\sqrt{2\pi}}(\hat{\phi}_k e^{ikx} + \hat{\phi}_{-k}e^{-ikx}) \tag{14.16}$$

$$= \frac{\Gamma(i\lambda)}{2\pi}\int_0^\infty dk\, k^{-i\lambda}(e^{-\pi\lambda/2}\hat{\phi}_k + e^{\pi\lambda/2}\hat{\phi}_{-k}). \tag{14.17}$$

where we used the following integral for $k>0$

$$\int_0^\infty \frac{dx}{x} x^{-i\lambda} e^{\pm ikx} = \Gamma(i\lambda)e^{\mp\pi\lambda/2}k^{-i\lambda}. \tag{14.18}$$

We can then compute $\langle \phi_{\lambda_1,\mathrm{I}}\phi_{\lambda_2,\mathrm{I}}\rangle$ directly from (14.11) and (14.17) with

$$\langle \phi_{\lambda_1,\mathrm{I}}\phi_{\lambda_2,\mathrm{I}}\rangle = \frac{\Gamma(i\lambda_1)}{2\pi}\frac{\Gamma(i\lambda_2)}{2\pi}\int_0^\infty dk_1\, k_1^{-i\lambda_1}\int_0^\infty dk_2\, k_2^{-i\lambda_2} \tag{14.19}$$
$$\times\Big(e^{-\pi(\lambda_1-\lambda_2)/2}\underbrace{\langle\phi_{k_1}\phi_{-k_2}\rangle}_{\frac{1}{2|k_1|}\delta(k_1-k_2)} + e^{\pi(\lambda_1-\lambda_2)/2}\underbrace{\langle\phi_{-k_1}\phi_{k_2}\rangle}_{\frac{1}{2|k_1|}\delta(k_1-k_2)}\Big)$$

$$= \frac{\Gamma(i\lambda_1)}{2\pi}\frac{\Gamma(i\lambda_2)}{2\pi}\cosh(\pi(\lambda_1-\lambda_2)/2)\int_0^\infty \frac{dk}{k}k^{-i(\lambda_1+\lambda_2)} \tag{14.20}$$

$$= \frac{|\Gamma(i\lambda_1)|^2}{4\pi^2}\cosh(\pi\lambda_1)(2\pi)\delta(\lambda_1+\lambda_2) \tag{14.21}$$

$$= \frac{1}{2|\lambda_1|}\coth(\pi|\lambda_1|)\delta(\lambda_1+\lambda_2) \tag{14.22}$$

where in the last line we used the identity

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)} \implies |\Gamma(i\lambda)|^2 = \frac{\pi}{\lambda\sinh(\pi\lambda)}. \tag{14.23}$$

We are now done. The expression

$$\boxed{\langle \phi_{\lambda_1,\mathrm{I}}\phi_{\lambda_2,\mathrm{I}}\rangle = \frac{1}{2|\lambda_1|}\coth(\pi|\lambda_1|)\delta(\lambda_1+\lambda_2)} \tag{14.24}$$

is identical to (14.11), save for the extra $\coth(\pi|\lambda|)$ factor analogous to the harmonic oscillator at finite temperature (14.5). In fact, from the above formula we can instantly read off $\beta = 2\pi$, which after we normalize properly for the time experienced by an observer with acceleration a becomes $\beta = 2\pi/a$, i.e. the inverse Unruh temperature!

In this interpretation of the Unruh effect, the role of quantum mechanics is simply to provide "noise" to the Gaussian ground state wave functional, which after a change of coordinates to an accelerating observer appears to be thermal. This is why the Unruh temperature is proportional to $\hbar$, as it is the constant $\hbar$ which sets the strength of the noise.

14.2 The ground state wavefunctional written with Rindler modes

We are now in a good position to give the exact expression for the wave functional in terms of Rindler modes.

The "wave functional" $\Psi[\phi]$ is the QFT generalization of the quantum mechanical wave function. $\Psi[\phi]$ assigns a complex number to every classical field configuration $\phi(x)$ on a $t = const$ timeslice.

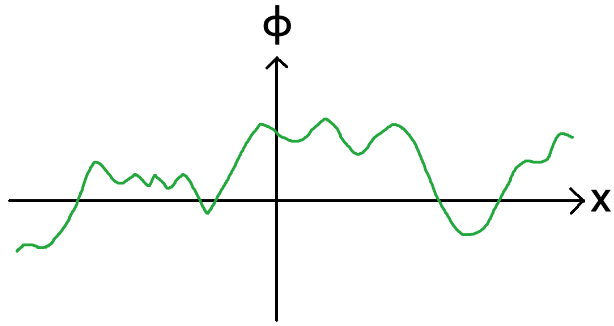

Figure 17: An example of a classical field configuration $\phi(x)$ we can plug into the ground state wave functional $\Psi_0[\phi]$ and get out a complex number.

The well-known expression for the Gaussian ground state wave functional written in terms of plane-wave modes ϕ_k is (where we have re-instated $\hbar$)

$$\Psi_0[\phi] \propto \exp\left(-\frac{1}{\hbar}\int_0^\infty dk\,|k|\phi_k\phi_{-k}\right). \tag{14.25}$$

Let's now show how to write the ground state wave functional written in terms of the Rindler field variables $\phi_{\lambda,\mathrm{I}}$ and $\phi_{\lambda,\mathrm{II}}$. Because we know that the wavefunctional must be some Gaussian combination of these field variables, we shall use the following fact about Gaussian integrals. If one defines the Gaussian integral with the complex coordinates z_i, $\bar{z}_i$ for $i = 1,\dots,n$

$$\int dz_1 d\bar{z}_1 \dots dz_n d\bar{z}_n \exp\left(-\sum_{i,j=1}^{n} z_i A_{ij}\bar{z}_j\right) = \frac{\pi^n}{\det A} \tag{14.26}$$

where the measure is defined to be $dz_i d\bar{z}_i \equiv d\mathrm{Re}(z_i)d\mathrm{Im}(z_i)$, then we can use the equation for the variation of the determinant

$$\frac{\partial}{\partial A_{ij}}\det(A) = \det(A)(A^{-1})_{ij} \tag{14.27}$$

to get

$$\langle z_i \bar{z}_j \rangle = \frac{\int dz_1 d\bar{z}_1 \dots dz_n d\bar{z}_n \exp\Big(-\sum_{i,j} z_i A_{ij} \bar{z}_j\Big) z_i \bar{z}_j}{\int dz_1 d\bar{z}_1 \dots dz_n d\bar{z}_n \exp\Big(-\sum_{i,j} z_i A_{ij} \bar{z}_j\Big)} \tag{14.28}$$

$$= (A^{-1})_{ij}. \tag{14.29}$$

So, in order to find the ground state wave functional in terms of Rindler modes, all we need to do is calculate all quadratic expectation values $\langle \phi_{\lambda_1,\mathrm{I}} \phi_{\lambda_2,\mathrm{I}} \rangle$, $\langle \phi_{\lambda_1,\mathrm{II}} \phi_{\lambda_2,\mathrm{II}} \rangle$, and $\langle \phi_{\lambda_1,\mathrm{I}} \phi_{\lambda_2,\mathrm{II}} \rangle$. While we have already written $\hat{\phi}_{\lambda,\mathrm{I}}$ in terms of $\hat{\phi}_k$ in (14.17), we need to do the same for $\hat{\phi}_{\lambda,\mathrm{II}}$. A computation with is analogous to the one we performed for $\hat{\phi}_{\lambda,\mathrm{I}}$ yields

$$\hat{\phi}_{\lambda,\mathrm{II}} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dx}{x} x^{-i\lambda} \hat{\phi}(-x) \tag{14.30}$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dx}{x} x^{-i\lambda} \int_0^\infty \frac{dk}{\sqrt{2\pi}} (\hat{\phi}_k e^{-ikx} + \hat{\phi}_{-k} e^{ikx}) \tag{14.31}$$

$$= \frac{\Gamma(i\lambda)}{2\pi} \int_0^\infty dk \; k^{-i\lambda} (e^{\pi\lambda/2} \hat{\phi}_k + e^{-\pi\lambda/2} \hat{\phi}_{-k}). \tag{14.32}$$

A similar computation to the one we performed to calculate $\langle \phi_{\lambda_1,\mathrm{I}} \phi_{\lambda_2,\mathrm{I}} \rangle$ in (14.19) can be used to calculate for $\langle \phi_{\lambda_1,\mathrm{I}} \phi_{\lambda_2,\mathrm{II}} \rangle$. The result is

$$\langle \phi_{\lambda_1,\mathrm{I}} \phi_{\lambda_2,\mathrm{II}} \rangle = \frac{1}{|\lambda_1|} \frac{1}{\sinh(\pi|\lambda_1|)} \delta(\lambda_1 + \lambda_2) \tag{14.33}$$

which is non-zero because there is entanglement between regions I and II.

All three possible pairings are then

$$\begin{aligned} \langle \phi_{\lambda_1,\mathrm{I}} \phi_{\lambda_2,\mathrm{I}} \rangle &= \frac{1}{2|\lambda_1|} \coth(\pi|\lambda_1|) \delta(\lambda_1 + \lambda_2) \\ \langle \phi_{\lambda_1,\mathrm{II}} \phi_{\lambda_2,\mathrm{II}} \rangle &= \frac{1}{2|\lambda_1|} \coth(\pi|\lambda_1|) \delta(\lambda_1 + \lambda_2) \\ \langle \phi_{\lambda_1,\mathrm{I}} \phi_{\lambda_2,\mathrm{II}} \rangle &= \frac{1}{|\lambda_1|} \frac{1}{\sinh(\pi|\lambda_1|)} \delta(\lambda_1 + \lambda_2) \end{aligned} \tag{14.34}$$

Using the above equations and (14.29), and noting that the above expectation values arise from integrating said variables against $|\Psi_0|^2$, we can write (restoring $\hbar$ again)

$$\boxed{\Psi_0[\phi] \propto \exp\left(-\frac{1}{\hbar} \int_0^\infty d\lambda \; \lambda \left[\frac{\phi_{\lambda,\mathrm{I}} \phi_{-\lambda,\mathrm{I}} + \phi_{\lambda,\mathrm{II}} \phi_{-\lambda,\mathrm{II}}}{\tanh(\pi\lambda)} - \frac{\phi_{\lambda,\mathrm{I}} \phi_{-\lambda,\mathrm{II}} + \phi_{-\lambda,\mathrm{I}} \phi_{\lambda,\mathrm{II}}}{\sinh(\pi\lambda)} \right] \right)} \tag{14.35}$$

If you would like to check the above equation for yourself, you'll need to use the matrix fact that

$$\begin{pmatrix} \coth(\pi\lambda) & 1/\sinh(\pi\lambda) \\ 1/\sinh(\pi\lambda) & \coth(\pi\lambda) \end{pmatrix}^{-1} = \begin{pmatrix} 1/\tanh(\pi\lambda) & -1/\sinh(\pi\lambda) \\ -1/\sinh(\pi\lambda) & 1/\tanh(\pi\lambda) \end{pmatrix}. \tag{14.36}$$

Let us now see what happens when we "integrate out" the region II modes from the wave functional. Because the wave functional is Gaussian, there is a trick we can use. Consider the 1d Gaussian integral

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/2a}. \tag{14.37}$$

The clever trick to compute the above integral (up to the square root prefactor out front) is to solve the value of x which maximizes the integrand and then simply plug it back in:

$$0 = \frac{\partial}{\partial x}\big(-ax^2 + bx\big) = -2ax + b \implies x = \frac{b}{2a} \tag{14.38}$$

$$(-ax^2 + bx)\Big|_{x=\frac{b}{2a}} = \frac{b^2}{4a} \tag{14.39}$$

which is exactly the exponent resulting from the integral in (14.37). It is similarly straightforward to show that this trick also works for complex Gaussian integrals.

Let us therefore compute the value of $\phi_{\lambda,\mathrm{II}}$ which maximizes the wavefunctional, given that we are holding all of the values of $\phi_{\lambda,\mathrm{I}}$ fixed:

$$0 = \frac{\delta}{\delta \phi_{\lambda,\mathrm{II}}} \Psi_0[\phi] = -\frac{\lambda}{\hbar}\left(\frac{\phi_{-\lambda,\mathrm{II}}}{\tanh(\pi\lambda)} - \frac{\phi_{-\lambda,\mathrm{I}}}{\sinh(\pi\lambda)}\right)\Psi_0[\phi] \tag{14.40}$$

$$\implies \phi_{\lambda,\mathrm{II}} = \frac{1}{\cosh(\pi\lambda)}\phi_{\lambda,\mathrm{I}}. \tag{14.41}$$

We will then plug this value back into the wave functional (squared) to integrate out all of the $\phi_{\lambda,\mathrm{II}}$'s up to a normalization constant:

$$\int \prod_{\lambda'>0} d\phi_{\lambda',\mathrm{II}} d\phi^*_{\lambda',\mathrm{II}} \Big|\Psi_0[\phi]\Big|^2 \propto \left(\Big|\Psi_0[\phi]\Big|^2\right)\Bigg|_{\phi_{\lambda,\mathrm{II}}=\frac{\phi_{\lambda\mathrm{I}}}{\cosh(\pi\lambda)}}. \tag{14.42}$$

The resulting expression is

$$\int \prod_{\lambda'>0} d\phi_{\lambda',\mathrm{II}} d\phi^*_{\lambda',\mathrm{II}} \Big|\Psi_0[\phi]\Big|^2 \propto \exp\left(-\frac{2}{\hbar}\int_0^\infty d\lambda\, \lambda \frac{\phi_{\lambda,\mathrm{I}}\phi_{-\lambda,\mathrm{I}}}{\coth(\pi\lambda)}\right). \tag{14.43}$$

14.3 Wavefunctional Intuition: explaining the temperature

14.3.1 Why temperature comes from entanglement

I think it's safe to say that the Unruh effect is very uninuitive. When I initially attempted to devise some sort of explanation for why it occurs, I hit a seemingly impenetrable roadblock: all of the treatments I could find on the Unruh effect crucially used the mathematical formalism of *creation and annihilation operators* in order to derive the effect. In particular, if one puts a positive frequency solution to the wave equation (for some notion of 'positive frequency') in one slot of the Klein Gordon inner product, and the field operator $\hat{\phi}$ in the other slot, one constructs a creation operator. Likewise, if one puts a negative frequency solution in that slot of the Klein Gordon inner product instead, one constructs an annihilation operator. If one can understand these positive and frequency "modes" (which are just solutions to the wave equation) pictorially, one can build up *some* intuition for how the Unruh effect works. However, I feel that any attempt at building intuition for the Unruh effect that leverages the creation and annihilation operators will never be 100% satisfactory. While the creation and annihilation operators are indispensable pieces of mathematical formalism, it is difficult to look at the definitions of these operators and derive much meaning from them. They seem to be an "irreducible" piece of mathematical technology, which cannot be understood intuitively in terms of more primitive objects.

One of the reasons why it is difficult to "understand" the Unruh effect is because different observers have different notions of what constitutes a "particle." The Unruh effect is a situation where quantum field theory is better understood using quantum *fields*, rather than particles.

Therefore, in order to build intuition for *why* the Unruh effect happens, we need to build a great degree of intuition for how the ground state wave functional $\Psi_0[\phi]$ behaves. I will warn you ahead of time that this section will devolve into a rather... elaborate analysis of the wave functional, (the logic of which will eventually culminate in section 14.3.6) but hey, you're the one reading it, so clearly you're into that kind of thing.

Before looking at the ground state wave functional in great detail, let's remind ourselves what a thermal distribution of positions looks like for the 1D oscillator, as opposed to the ground state. As discussed in the last section, a thermal distribution multiplies the size of the position variance for an oscillator of frequency ω by a factor of $\coth^{1/2}(\beta\hbar\omega/2)$:

$$\langle 0|\,\hat{q}^2\,|0\rangle = \frac{\hbar}{2m\omega}, \qquad \langle q^2\rangle_\beta = \frac{\hbar}{2m\omega}\coth(\beta\hbar\omega/2). \tag{14.44}$$

If the frequency ω is very large, note that $\coth(\beta\hbar\omega/2) \to 1$ and the probability distribution isn't changed very much by the introduction of a non-zero temperature. This makes sense, because at high frequencies there is a large energy gap of that mode between the ground state and the first excited state. (High frequency degrees of freedom stay essentially "frozen out" at finite temperature.) However, for very *low* frequencies ω, we instead have $\coth(\beta\hbar\omega/2) \to \infty$ and the expected variance of field values grows extremely large.

Jumping to quantum field theory, where our field is a collection of an infinite number of harmonic oscillators of different frequencies, we can therefore see that the signature of thermality is exactly this: for *high* frequency modes, the variance in the field values stays at the low (zero occupation number) value, but at *low* frequencies, the variance in the field values get very large. This is the essential feature of thermality that we will attempt to explain.

So, how does this arise in our wave functional? Well, we have written our wave functional in terms of Rindler modes $\phi_{\lambda,\mathrm{I}}$ and $\phi_{\lambda,\mathrm{II}}$ (which are restricted in regions I and II on the $t = 0$ slice, respectively) in Equation (14.35). Notice that the field variables $\phi_{\lambda,\mathrm{I}}$ and $\phi_{\lambda,\mathrm{II}}$ are actually complex field variables, and to write our wave functional as a function of real variables, we must use the real and imaginary parts, i.e.

$$\phi_{\lambda,\mathrm{I}} = \mathrm{Re}(\phi_{\lambda,\mathrm{I}}) + i\,\mathrm{Im}(\phi_{\lambda,\mathrm{I}}) \tag{14.45}$$

$$\phi_{\lambda,\mathrm{II}} = \mathrm{Re}(\phi_{\lambda,\mathrm{II}}) + i\,\mathrm{Im}(\phi_{\lambda,\mathrm{II}}). \tag{14.46}$$

We can simply plug the above equations into Eq. (14.35) to express our wavefunctional as a function of real field modes. The real and imaginary parts end up factoring, and it is possible to just look at, say, the slice of the wave functional which depends on $\text{Re}(\phi_{\lambda,\text{I}})$ and $\text{Re}(\phi_{\lambda,\text{II}})$ for a particular λ, as it will be independent from all the other field variables.

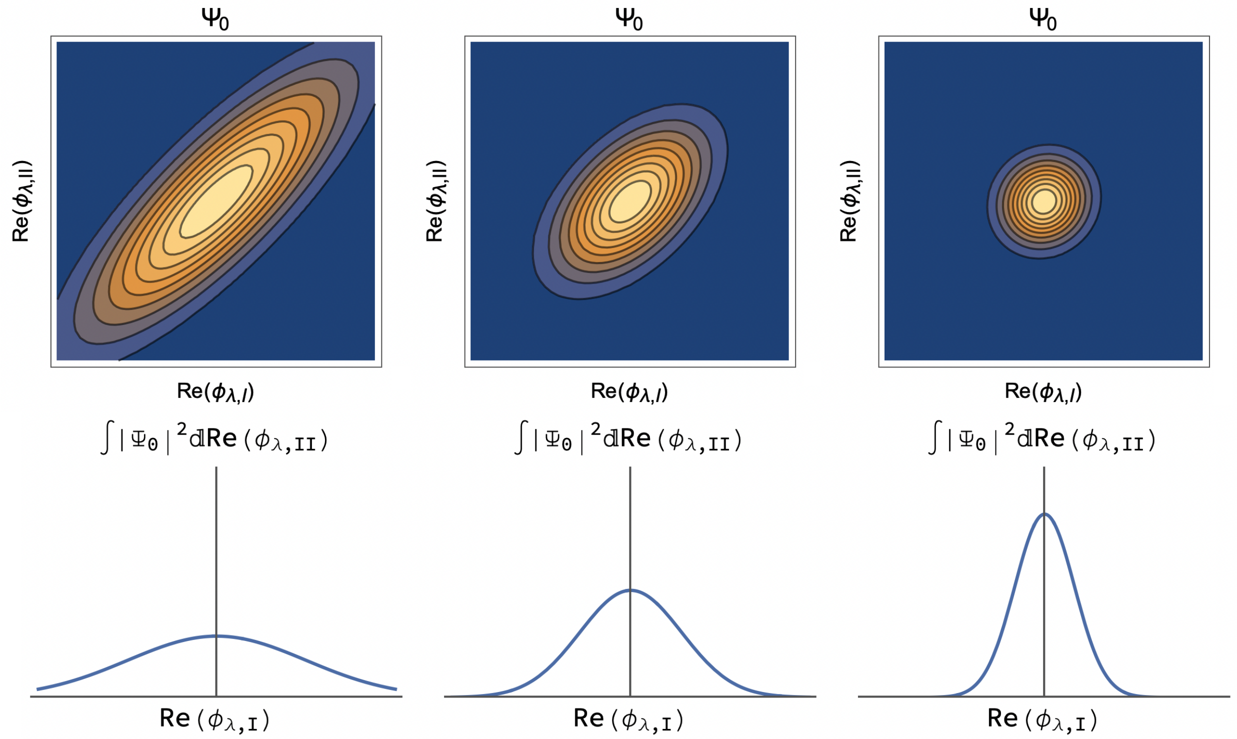

Figure 18: Here we graph a slice of the ground state wave functional Ψ_0 as it depends on the variables $\text{Re}(\phi_{\lambda,\text{I}})$ and $\text{Re}(\phi_{\lambda,\text{II}})$, for $\lambda = 0.2$, $\lambda = 0.4$, and $\lambda = 1.0$ respectively. As λ increases (Rindler frequency increases) the wavefunction becomes less and less squeezed and less entangled, and approaches the ground state for that 2-dimensional oscillator. Underneath, we plot the result of integrating out the $\phi_{\lambda,\text{II}}$ degree of freedom from $|\Psi_0|^2$. Note that the more squeezed/entangled the state is, the larger the "spread" in the values of $\phi_{\lambda,\text{I}}$.

If we then graph $\Psi_0(\text{Re}(\phi_{\lambda,\text{I}}), \text{Re}(\phi_{\lambda,\text{II}}))$ for a particular λ, keeping all other field variables fixed, we get Figure 18. In Figure 18, notice that the smaller λ is (referring to modes with low Rindler frequency) the more entangled the state is between the $\phi_{\lambda,\text{I}}$ variable and $\phi_{\lambda,\text{II}}$ variable: the wave function is more "squeezed," and the value of one variable depends heavily on the value of the other variable. That is, $\text{Re}(\phi_{\lambda,\text{I}})$ and $\text{Re}(\phi_{\lambda,\text{II}})$ are very likely to be equal to each other if λ is low. However, if λ is large, then the wave function becomes much less squeezed, approaching the lowest occupation number state for those two modes.

Now, of course, an accelerating observer in region I will only every be able to observe the I modes. Therefore, if we integrate out the II modes from $|\Psi_0|^2$, we get the probability of observing $\text{Re}(\phi_{\lambda,\text{I}})$ at each possible value. After this integration, we see that the highly squeezed low λ field variables have a very spread out distribution, while the non-squeezed high λ field modes are much more sharply peaked. But this is exactly the behavior we said was characteristic of thermality, that low frequency field modes should have more spread out distributions, and high frequency field modes should have less spread out distributions. In fact, thinking about it a little harder, one can argue that this sort of "squeezing" of the modes is the *only* way we can get this "spreading out" behavior in $\mathrm{Re}(\phi_{\lambda,\mathrm{I}})$ which is symmetric between I and II and consistent with the mode's Hamiltonian of frequency λ.

In fact, as discussed in section 14.2, if we solve for the value of $\phi_{\lambda,\mathrm{II}}$ that maximizes $|\Psi_0|^2$ given a fixed $\phi_{\lambda,\mathrm{I}}$, that value is given by

$$\phi_{\lambda,\mathrm{II}} = \frac{1}{\cosh(\pi\lambda)}\phi_{\lambda,\mathrm{I}} \tag{14.47}$$

which means $\phi_{\lambda,\mathrm{II}} = \phi_{\lambda,\mathrm{I}}$ as $\lambda \to 0$ and $\phi_{\lambda,\mathrm{II}} = 0$ as $\lambda \to \infty$. This is just a way to restate that $\phi_{\lambda,\mathrm{I}}$ and $\phi_{\lambda,\mathrm{II}}$ are highly correlated for low λ but uncorrelated at high λ.

So, is it possible for us to build intuition for WHY the mode variables $\phi_{\lambda,\mathrm{I}}$ and $\phi_{\lambda,\mathrm{II}}$ are highly correlated for small λ, but essentially independent for large λ? This is where the "explaination" truly begins (and things start to get a bit sweaty).

14.3.2 Why boost modes are bigger on one half than the other

Rindler modes, are, of course, eigenfunctions of the boost vector field

$$K = x\partial_t + t\partial_x = -u\partial_u + v\partial_v. \tag{14.48}$$

These functions roughly look like $e^{i\lambda\log(u)}$ and $e^{i\lambda\log(v)}$. On a constant time slice $t = 0$, we have $v = -u = x$. The rindler modes then reduce to $e^{i\lambda\log(x)}$ (where we are not being careful with the $i\epsilon$ prescription in the log right now). Taking a hard look at the functions $e^{i\lambda\log(x)}$, we notice that they are eigenfunctions of the scale transformation

$$\text{scale transformation:} \;\; f(x) \to f(e^{\theta}x). \tag{14.49}$$

Infinitesimally, this means that the Rindler modes are eigenfunctions of the scale transformation generator $x\partial_x$ on the $t = 0$ slice. So, comparing inertial plane wave modes e^{ikx} with accelerating Rindler modes, we see that the plane waves are eigenfunctions of the translation operator while the Rindler modes are eigenfunctions of the scale operator.

$$\text{plane waves diagonalize momentum :} \qquad (\partial_x)e^{ikx} = ike^{ikx} \tag{14.50}$$

$$\text{Rindler modes diagonalize scaling :} \qquad (x\partial_x)x^{i\lambda} = i\lambda x^{i\lambda}. \tag{14.51}$$

So, on a $t = 0$ slice, how can you make a Rindler mode of Rindler energy λ as a sum of plane waves? The answer is to average over a plane wave subjected to all possible scale transformations, where each scale is weighted by a certain λ-dependent phase. In particular, this average is given by

$$\Phi_{>,\lambda}(x) = \frac{1}{\Gamma(-i\lambda)}\int_{-\infty}^{\infty} d\theta e^{-i\lambda\theta}e^{ie^{\theta}x}. \tag{14.52}$$

Here, θ parameterizes the scale transformation of $x \to e^{\theta}x$, and $e^{ie^{\theta}x}$ is the correspondingly scaled plane wave. $\Gamma(-i\lambda)$ is a conveniant normalization constant. The factor $e^{-i\lambda\theta}$ is the aforementioned λ dependent phase factor that makes the whole expression an eigenfunction of $x\partial_x$ with eigenvalue $i\lambda$. To see why this average is an eigenfunction of scaling, note the algebra

$$(x\partial_x)\int_{-\infty}^{\infty} d\theta e^{-i\lambda\theta} e^{ie^{\theta}x} = \int_{-\infty}^{\infty} d\theta e^{-i\lambda\theta}\left(\partial_\theta\right) e^{ie^{\theta}x} \tag{14.53}$$

$$= -\int_{-\infty}^{\infty} d\theta(\partial_\theta e^{-i\lambda\theta}) e^{ie^{\theta}x} \tag{14.54}$$

$$= i\lambda \int_{-\infty}^{\infty} d\theta e^{-i\lambda\theta} e^{ie^{\theta}x}. \tag{14.55}$$

Notice that $\Phi_{>,\lambda}$, which by (14.52) is a sum of functions like $e^{ie^{\theta}x}$, only has $k > 0$ modes in its Fourier transform because $e^{\theta} > 0$. We can also define a companion function $\Phi_{>,\lambda}$ which only has $k < 0$ modes in its Fourier transform:

$$\Phi_{<,\lambda}(x) = \frac{1}{\Gamma(-i\lambda)}\int_{-\infty}^{\infty} d\theta e^{-i\lambda\theta} e^{-ie^{\theta}x}. \tag{14.56}$$

Both $\Phi_{<,\lambda}(x)$ and $\Phi_{<,\lambda}(x)$ have an eigenvalue of $i\lambda$ under scaling.

$$(x\partial_x)\Phi_{<,\lambda}(x) = i\lambda\Phi_{<,\lambda}(x), \qquad (x\partial_x)\Phi_{>,\lambda}(x) = i\lambda\Phi_{>,\lambda}(x). \tag{14.57}$$

They are related by complex conjugation as

$$\Phi^*_{>,\lambda}(x) = \Phi_{<,-\lambda}(x). \tag{14.58}$$

Now that we've motivated the definitions of the modes $\Phi_{>,\lambda}$ and $\Phi_{<,\lambda}$, let's evaluate their integrals to express them as functions of x. (In order for the integral to converge, we must add a small imaginary part to x.) We find for $\Phi_{>,\lambda}$

$$\Phi_{>,\lambda}(x) = e^{\pi\lambda/2}(x + i\epsilon)^{i\lambda} \tag{14.59}$$

$$= (e^{\pi\lambda/2}x^{i\lambda}\Theta(x) + e^{-\pi\lambda/2}(-x)^{i\lambda}\Theta(-x)). \tag{14.60}$$

Interestingly, $\Phi_{>,\lambda}$ is $e^{\pi\lambda}$ times *bigger* in the $x > 0$ region than the $x < 0$ region. Likewise, for $\Phi_{<,\lambda}$

$$\Phi_{<,\lambda}(x) = e^{-\pi\lambda/2}(x - i\epsilon)^{i\lambda} \tag{14.61}$$

$$= (e^{-\pi\lambda/2}x^{i\lambda}\Theta(x) + e^{\pi\lambda/2}(-x)^{i\lambda}\Theta(-x)). \tag{14.62}$$

and we find that $\Phi_{<,\lambda}$ is $e^{-\pi\lambda}$ *smaller* in the $x > 0$ region than the $x < 0$ region.

Let us now meditate on why $\Phi_{>,\lambda}$ and $\Phi_{<,\lambda}$ have different magnitudes in the two halves of space.

Ultimately, it has to do with destructive interference and constructive interference. Say you are going to add two complex numbers together, like $e^{i\theta_1}$ and $e^{i\theta_2}$. The sum will have the largest magnitude if the angles θ_1 and θ_2 are equal. If we add a *bunch* of complex numbers together, then the sum will be greatest if all of the angles we are summing over are close together.

With this in mind, let's take a close look at the integral (14.52) and get a feel for how the integrand behaves.

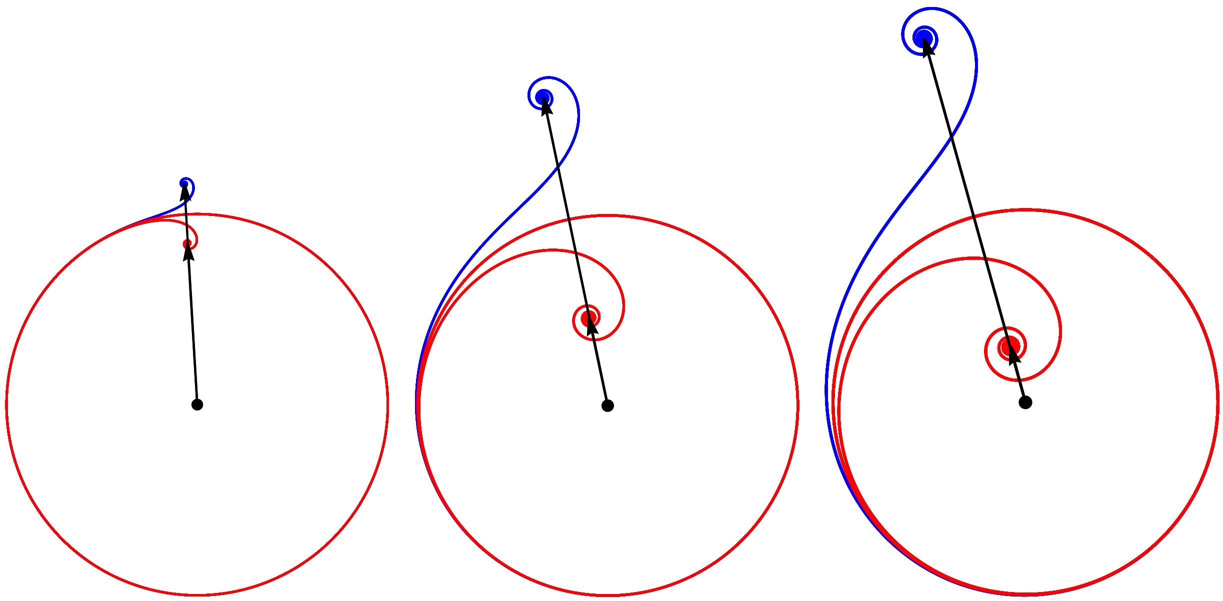

Figure 19: We draw the results of numerical integration on the complex plane of (14.52) for $x > 0$ (blue) and $x < 0$ (red) for $\lambda = 0.1$, $\lambda = 0.4$, $\lambda = 0.6$, respectively. The black dot in the center is the origin, and the final result of the integration is highlighted by drawing a vector from the origin to the final point. For θ a large negative number, the integrand is oscillatory which is accounted for by placing the origin in the center of the circle. Notice that the $x > 0$ integral ends with a larger magnitude than the $x < 0$ integral because when $x > 0$ there is a region where the phase of the summed complex numbers are stationary, thus allowing the sum to get further from the origin. The $x > 0$ sum ends up being a factor of $e^{\pi\lambda}$ further from the origin than the $x < 0$ sum. Note also that both sums are collinear with the origin.

More specifically, let's look at the integrand in (14.52) in three zones: where θ is a large negative number, where θ is a large positive number, and where θ is in the middle. If θ is a large negative number, then $e^\theta \approx 0$ and the integrand is essentially just $e^{-i\lambda\theta}$ and is completely oscillatory. Such an oscillatory integral averages to 0. If θ is a large positive number, then e^θ is very large and rapidly growing, so $e^{ie^\theta x}$ is a rapidly oscillating phase which destructively inteferes to zero. Now, the behavior of the phase

$$\exp\left(-i\lambda\theta + ie^\theta x\right)$$

behaves very differently for intermediate θ depending on if $x > 0$ or $x < 0$. For simplicity, let's assume that $\lambda > 0$, although ultimately both signs of λ are actually relevant. If $x > 0$, then at $e^\theta/\theta = \lambda/x$ the phases are stationary and constructively add together to make the magnitude of the sum larger. If $x < 0$, then there is no θ for which the phase is stationary, and thus the entire sum is not as large as it would be for $x > 0$. (This behavior is flipped for $\lambda < 0$.)

The two integrals, for $x > 0$ and $x < 0$, are depicted in Figure 19 for three different values of λ. The blue swirl depicts the numerically integrated sum for $x > 0$ and the red swirl depicts the numerically integrated sum for $x < 0$. The black dot it the origin. Note that the blue swirl always ends up further from the origin than the red swirl. In fact, the blue swirl ends up to be a factor of $e^{\pi\lambda}$ further from the origin than the red swirl.

(We should of course also ask why the blue swirl ends more far from the red swirl the larger λ is. This is easier to see if you rescale θ as $\theta' = \theta\lambda$, making the phases being summed $\exp\left(-i\theta' + i e^{\theta'/\lambda} x\right)$. Then we can see that the larger λ is, the more slowly the exponentially growing phase grows and the longer amount of time the phases are near stationary when $x > 0$.)

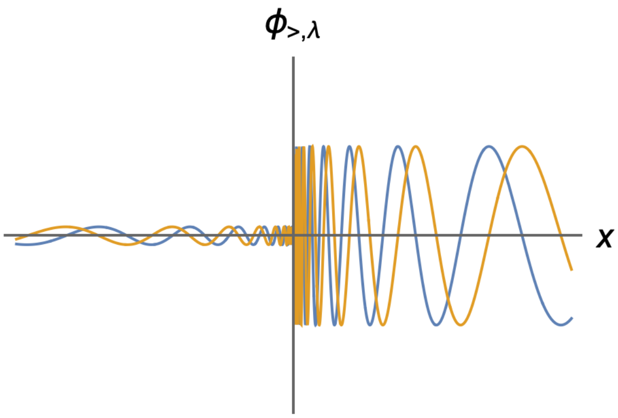

Figure 20: $\phi_{>,\lambda}$ is the coefficient of the mode $e^{\pi\lambda/2}(x+i\epsilon)^{i\lambda}$

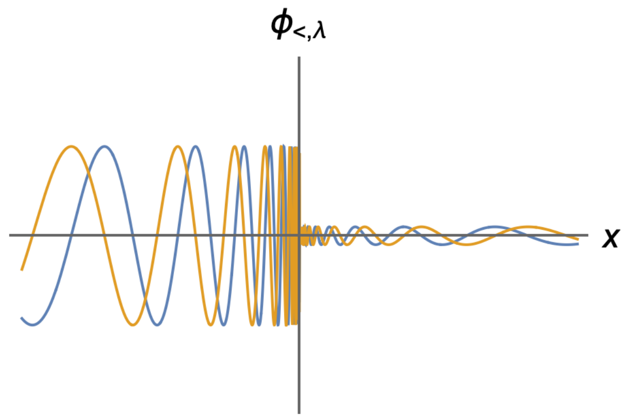

Figure 21: $\phi_{<,\lambda}$ is the coefficient of the mode $e^{-\pi\lambda/2}(x-i\epsilon)^{i\lambda}$

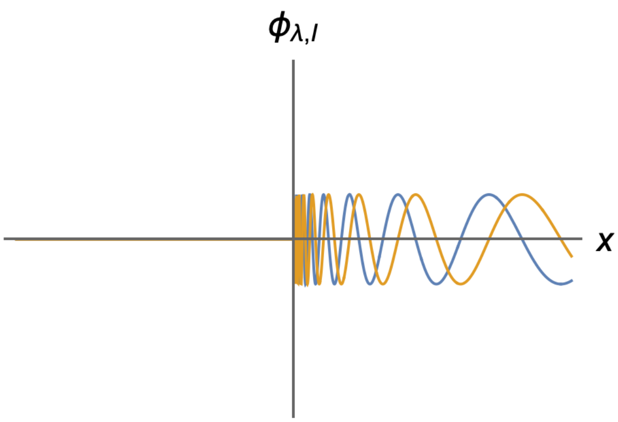

Figure 22: $\phi_{\lambda,\mathrm{I}}$ is the coefficient of the mode $\Theta(x)x^{i\lambda}$

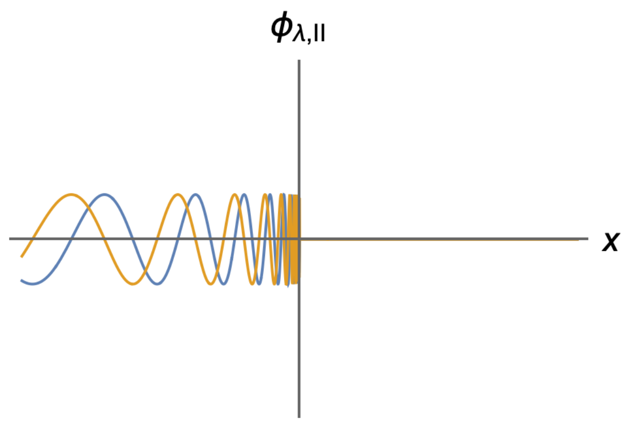

Figure 23: $\phi_{\lambda,\mathrm{II}}$ is the coefficient of the mode $\Theta(-x)(-x)^{i\lambda}$

14.3.3 Rewriting wave functional with these boost modes

Let us now rewrite the ground state wave functional of (14.25), reproduced below as

$$\Psi_0[\phi] \propto \exp\left(-\frac{1}{\hbar}\int_0^\infty dk\, |k| \phi_k \phi_{-k}\right). \tag{14.63}$$

in terms of our global boost modes $\Phi_{>,\lambda}(x)$ and $\Phi_{<,\lambda}(x)$. (We call them 'global' because they have support on both the left and right halves.) Towards this end, let us decompose the input field $\phi(x)$ as

$$\phi(x) = \int_{-\infty}^{\infty} \frac{d\lambda}{\sqrt{2\pi}} \left(\phi_{>,\lambda}\Phi_{>,\lambda}(x) + \phi_{<,\lambda}\Phi_{<,\lambda}(x)\right). \tag{14.64}$$

This should be compared to the standard plane wave decomposition

$$\phi(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \phi_k e^{ikx}. \tag{14.65}$$

Note that $\Phi_{>,\lambda}(x)$ and $\Phi_{<,\lambda}(x)$ are functions of x, while $\phi_{>,\lambda}$ and $\phi_{<,\lambda}$ are the numerical coefficients multiplying these functions. The relationship between $\phi_{>,\lambda}$ and $\Phi_{>,\lambda}(x)$ is the same as the relationship between ϕ_k and e^{ikx}. We can also decompose $\phi(x)$ in terms of the restricted Rindler modes as

$$\phi(x) = \int_{-\infty}^{\infty} \frac{d\lambda}{\sqrt{2\pi}} \left(\phi_{\lambda,\mathrm{I}}\Theta(x)x^{i\lambda} + \phi_{\lambda,\mathrm{II}}\Theta(-x)(-x)^{i\lambda}\right). \tag{14.66}$$

To figure out what the expression for Ψ_0 is in terms of $\phi_{>,\lambda}$ and $\phi_{<,\lambda}$ is, we could simply find a way to convert between ϕ_k and $\phi_{>,\lambda}$, $\phi_{<,\lambda}$ and simply plug that into (14.63). While we will do this later, it is more satisfying to justify what Ψ_0 must look like on very general grounds. In particular, we're going to argue that it must be a Gaussian where $\phi_{>,\lambda}$ must be paired with $\phi_{<,-\lambda}$.

The first consideration is that, because Ψ_0 is invariant under boosts (as the vacuum is Lorentz invariant), the expression for Ψ_0 cannot pair together (or entangle) ϕ's of different λ. To see why this is, consider the toy example of two harmonic oscillators with two unequal frequencies ω_1 and ω_2. If the joint wave function $\psi(q_1, q_2, t)$ evolves in such a way that $|\psi(q_1, q_2, t)|^2 = |\psi(q_1, q_2, 0)|^2$, i.e. that the probability of observing the particles in particular positions is invariant under time evolution, then ψ must be a product state of energy eigenstates in q_1 and q_2 respectively, i.e.

$$\psi(q_1, q_2, t) = \psi_{n_1}(q_1)\psi_{n_2}(q_2)e^{-i\omega_1(n_1+1/2)t}e^{-i\omega_2(n_2+1/2)t}.$$

If ψ did not factor as a product state in this way, then the interference cross-term between the differing frequencies would lead $|\psi|^2$ to be time dependent. Just as the time-invariance of this joint harmonic oscillator requires that the state be a product of energy eigenstates, so too does the boost invariance of Ψ_0 require that Ψ_0 must also be a product state of modes of differing Rindler frequency λ. (Actually, more than the modulus squared being time invariant, it is actually the wave function Ψ_0 itself which is boost invariant. Not even a phase is acquired under boosts. This is possible because there are modes of both positive λ and negative λ in equal proportion.)

In addition to the boost invariance of Ψ_0, there is another property of Ψ_0 which we can use to determine the wave function, namely that in (14.63), we can see that Fourier coefficients with $k > 0$ are paired with Fourier coefficients with $k < 0$ as $\phi_k\phi_{-k}$. Because $\Phi_{>,\lambda}$ is exclusively made up of Fourier coefficients with $k > 0$ and $\Phi_{<,\lambda}$ is exclusively made up of Fourier coefficients with $k < 0$, we know that Ψ_0 will likewise only pair up $\phi_{>,\lambda}$ with $\phi_{<,-\lambda}$. (Notice that $\phi^*_{-k} = \phi_k$ and $\phi^*_{>,\lambda} = \phi_{<,-\lambda}$. The fact that Ψ_0 only contains the combination $\phi_k\phi_{-k}$ comes from the general fact that energy eigenstates have wavefunctions which can always be chosen to be real.) Bringing this all together, we have

$$\Psi_0[\phi] \propto \exp\left(-\frac{2\pi}{\hbar}\int_{-\infty}^{\infty}\frac{d\lambda}{|\Gamma(i\lambda)|^2}\phi_{>,\lambda}\phi_{<,-\lambda}\right). \tag{14.67}$$

The general form of (14.67), namely that it is simply an exponential of an integral over the diagonal combination $\phi_{>,\lambda}\phi_{<,-\lambda}$, is as we have argued determined purely by the boost invariance, reality condition of Ψ_0, and the Gaussian nature of Ψ_0. To get the specific numerical prefactor in the integral, one must do a bit of actual math. In particular, one can use the formula

$$\phi_k = \frac{1}{\Gamma(-i\lambda)} \int_{-\infty}^{\infty} d\lambda (\Theta(k) k^{-1-i\lambda} \phi_{>,\lambda} + \Theta(-k)(-k)^{-1-i\lambda} \phi_{<,\lambda}) \tag{14.68}$$

which relates ϕ_k to $\phi_{>,\lambda}$ and $\phi_{<,\lambda}$, and plug the above expression into (14.63).

14.3.4 Getting "Real" with the wave functional

Our real scalar field $\phi(x)$, is of course, real. While the Fourier modes ϕ_k are not real, they obey a reality condition $\phi_k^* = \phi_{-k}$. Likewise, our global Rindler modes obey the reality condition $\phi_{>,\lambda}^* = \phi_{<,-\lambda}$. When thinking about the probabilistic correlation of variables, I find it very confusing to think about variables that are complex. Therefore, let's rephrase everything in terms of real variables:

$$\phi_{>,\lambda} = \text{Re}(\phi_{>,\lambda}) + i\,\text{Im}(\phi_{>,\lambda}) \tag{14.69}$$

$$\phi_{<,-\lambda} = \text{Re}(\phi_{>,\lambda}) - i\,\text{Im}(\phi_{>,\lambda}). \tag{14.70}$$

Plugging the above equations into (14.67), we find

$$\Psi_0[\phi] \propto \exp\left(-\frac{2\pi}{\hbar} \int_{-\infty}^{\infty} \frac{d\lambda}{|\Gamma(i\lambda)|^2} \left(\text{Re}(\phi_{>,\lambda})^2 + \text{Im}(\phi_{>,\lambda})^2 \right) \right) \tag{14.71}$$

Notice, then, that the probability distributions of the real and imaginary parts of $\phi_{>,\lambda}$ are independent from each other in the ground state. This is a crucial property of the ground state which is worth understanding a bit more deeply.

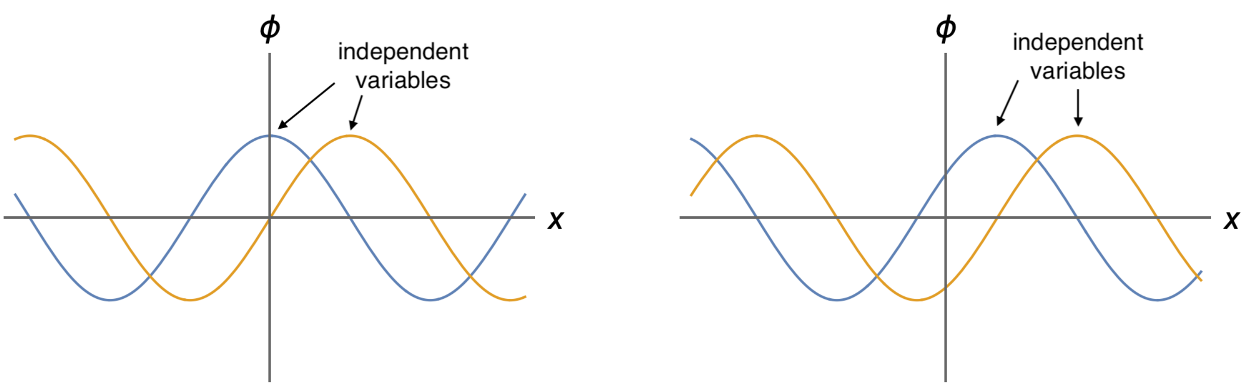

Figure 24: The amplitude of a sine wave and a cosine wave are independent uncorrelated variables in the ground state Ψ_0. Likewise, if we translate the sine wave and the cosine wave by a fixed amount, the amplitudes of the translated sine and cosine waves are also independent uncorrelated variables in the ground state.

The first thing to understand is that if we write $\phi(x)$ as a linear combination of sine waves and cosine waves, then the amplitudes of the sine Fourier modes and the cosine Fourier modes are uncorrelated independent variables in ground state wave functional Ψ_0. Furthermore, by translational invariance, if we shift the sine wave and the cosine wave by some amount in x, then the corresponding Fourier modes will still be independent variables. This is depicted in Figure 24.

From this fact, notice that the real and imaginary parts of our global Rindler mode are similarly just linear combinations of cosine and sine waves, respectively, that are shifted by frequency-dependant phases $-\lambda\theta$.

$$\begin{aligned} \mathrm{Re}(\Phi_{>,\lambda}(x)) &= \frac{1}{|\Gamma(i\lambda)|}\int_{-\infty}^{\infty} d\theta \cos\big(-\lambda\theta + e^{\theta}x - \arg[\Gamma(i\lambda)]\big) \\ \mathrm{Im}(\Phi_{>,\lambda}(x)) &= \frac{1}{|\Gamma(i\lambda)|}\int_{-\infty}^{\infty} d\theta \sin\big(-\lambda\theta + e^{\theta}x - \arg[\Gamma(i\lambda)]\big) \end{aligned} \tag{14.72}$$

Because the amplitudes of the shifted cosines and sines can each be regarded as uncorrelated independent variables, so too can their linear combinations $\mathrm{Re}(\phi_{>,\lambda})$ and $\mathrm{Im}(\phi_{>,\lambda})$. This is depicted in Figure 25.

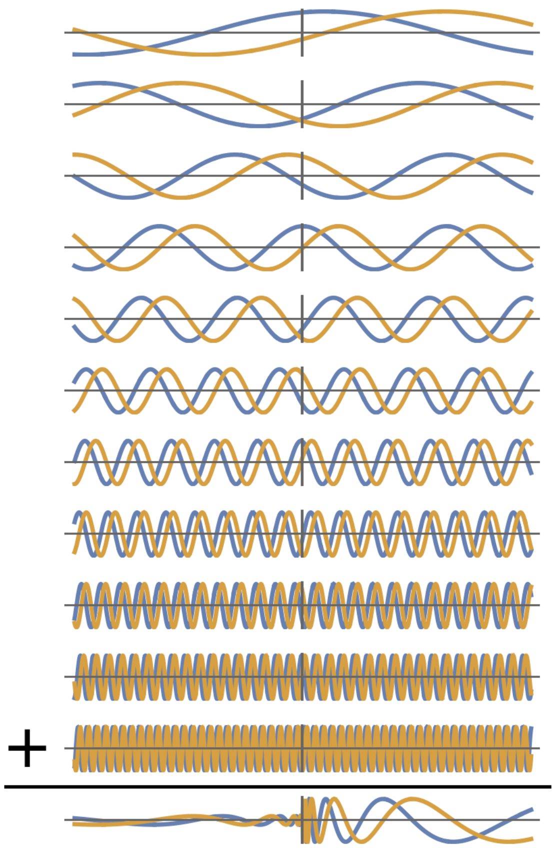

Figure 25: Here we plot the real and imaginary parts of the Fourier modes in (14.52) as functions of x with a wavelength dependent phase shift. The infinite sum of all of these Fourier modes is the bottom figure, which is the global Rindler mode with positive λ. Note that the Fourier modes add constructively in the $x > 0$ region and add destructively in the $x < 0$ region, such that the $x > 0$ region has a bigger amplitude. (If λ were negative this would be flipped.) Because amplitudes of the real and imaginary parts of the Fourier modes (cosines and sines, or blue and yellow in the adjacent plots) are independent variables with respect to the ground state wave functional Ψ_0, then the sums of these variables, i.e. the real and imaginary parts of the global Rindler modes are also independent variables with respect to Ψ_0.

14.3.5 How entanglement arises via maximization of the wave functional

Having expressed the ground state wave functional as a Gaussian of the uncorrelated real and imaginary parts of the Rindler modes in (14.71), we are ready to see how entanglement of the right and left halves of space arises via the maximization of the wave functional.

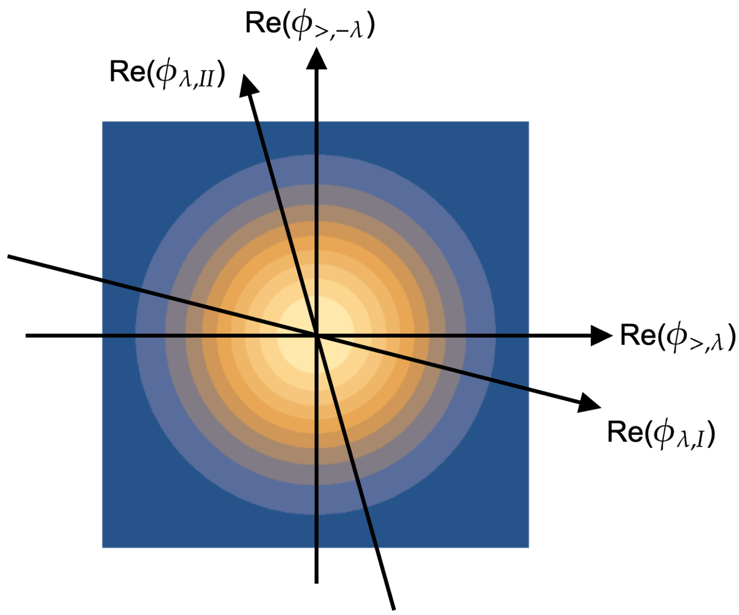

Figure 26: Here we present a contour plot of the ground state wave function Ψ_0. We present two axes: the Re($\phi_{>,\lambda}$) and Re($\phi_{>,-\lambda}$) axes, as well as the Re($\phi_{\lambda,\mathrm{I}}$) and Re($\phi_{-\lambda,\mathrm{II}}$) axes. With regards to the former axes, the wave functional is a Gaussian of circular profile, meaning that the distribution of Re($\phi_{>,\lambda}$) and Re($\phi_{>,-\lambda}$) are uncorrelated. For the second set of 'skew' axes, the corresponding variables are not uncorrelated.

Comparing the global mode coefficients $\phi_{>,\lambda}$ and $\phi_{<,\lambda}$ with the restricted mode coefficients $\phi_{\lambda,\mathrm{I}}$ and $\phi_{\lambda,\mathrm{II}}$ from (14.64) and (14.66), these two sets of coefficients are related to each other by

$$\phi_{\lambda,\mathrm{I}} = e^{\pi\lambda/2}\phi_{>,\lambda} + e^{-\pi\lambda/2}\phi_{<,\lambda} \tag{14.73}$$
$$\phi_{\lambda,\mathrm{II}} = e^{-\pi\lambda/2}\phi_{>,\lambda} + e^{\pi\lambda/2}\phi_{<,\lambda}. \tag{14.74}$$

Taking the real part of the above equation (we just as easily could take the imaginary part), we get

$$\mathrm{Re}\left(\phi_{\lambda,\mathrm{I}}\right) = e^{\pi\lambda/2}\,\mathrm{Re}(\phi_{>,\lambda}) + e^{-\pi\lambda/2}\,\mathrm{Re}(\phi_{>,-\lambda}) \tag{14.75}$$
$$\mathrm{Re}\left(\phi_{\lambda,\mathrm{II}}\right) = e^{-\pi\lambda/2}\,\mathrm{Re}(\phi_{>,\lambda}) + e^{\pi\lambda/2}\,\mathrm{Re}(\phi_{>,-\lambda}) \tag{14.76}$$

where we have used the relation $\mathrm{Re}(\phi_{<,\lambda}) = \mathrm{Re}(\phi_{>,-\lambda})$. We shall also record the inverse relations

$$\mathrm{Re}(\phi_{>,\lambda}) = \frac{1}{2\sinh(\pi\lambda)}\big(+ e^{\pi\lambda/2}\,\mathrm{Re}(\phi_{\lambda,\mathrm{I}}) - e^{-\pi\lambda/2}\,\mathrm{Re}(\phi_{\lambda,\mathrm{II}})\big) \tag{14.77}$$
$$\mathrm{Re}(\phi_{>,-\lambda}) = \frac{1}{2\sinh(\pi\lambda)}\big(- e^{-\pi\lambda/2}\,\mathrm{Re}(\phi_{\lambda,\mathrm{I}}) + e^{\pi\lambda/2}\,\mathrm{Re}(\phi_{\lambda,\mathrm{II}})\big). \tag{14.78}$$

Recall that the ground state wavefunctional Ψ_0, when written in terms of the real global Rindler mode coefficients $\mathrm{Re}(\phi_{>,\lambda})$ and $\mathrm{Im}(\phi_{>,\lambda})$, as in (14.71), is a product distribution of the real and imaginary parts of $\phi_{>,\lambda}$ for each λ individually. This means that if one were to graph $|\Psi_0|^2$ as a function of, say, $\text{Re}(\phi_{>,\lambda})$ and $\text{Re}(\phi_{>,-\lambda})$, then it would be a Gaussian with with lines of constant $|\Psi_0|^2$ being circles centered at the origin. This is drawn in Figure 26.

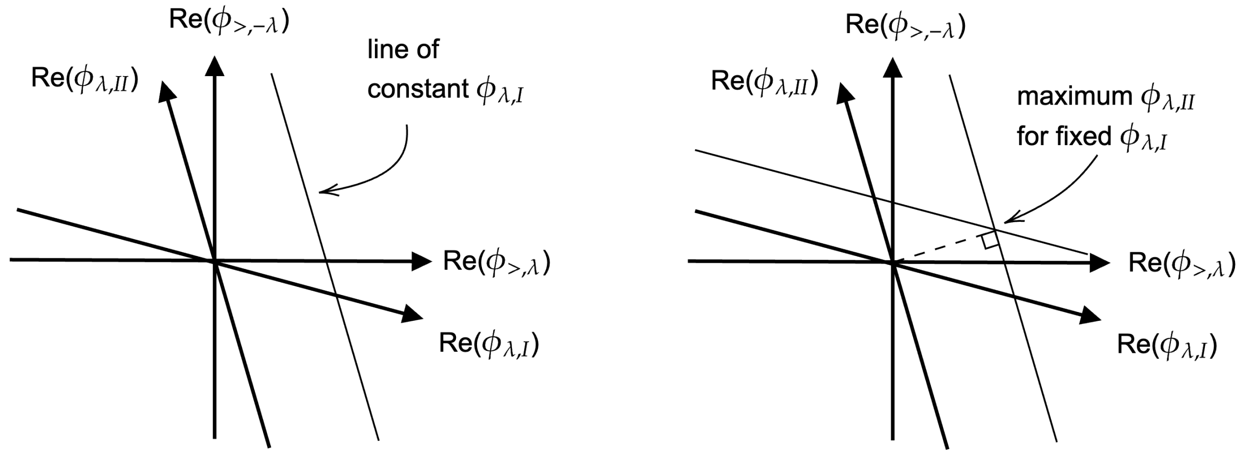

Figure 27: Here we explain how to find the value of $\phi_{\lambda,\text{II}}$ which maximizes $|\Psi_0|^2$ given a fixed value of $\phi_{\lambda,\text{I}}$. Moving along a line of fixed $\phi_{\lambda,\text{I}}$, $|\Psi_0|^2$ is maximized at the point on the line which is the closest distance to the origin.

While the $\text{Re}(\phi_{>,\lambda})$ and $\text{Re}(\phi_{>,-\lambda})$ axes appear orthogonal in Figure 26, the $\text{Re}(\phi_{\lambda,\text{I}})$ and $\text{Re}(\phi_{\lambda,\text{II}})$ axes appear skewed, as given by (14.75) and (14.76). The slope of the $\text{Re}(\phi_{\lambda,\text{I}})$ and $\text{Re}(\phi_{\lambda,\text{II}})$ axes are $-e^{-\pi\lambda}$ and $-e^{\pi\lambda}$, respectively.

So, how then can we understand the entanglement between $\phi_{\lambda,\text{I}}$ and $\phi_{\lambda,\text{II}}$ using the above image? Well, remember that the essential feature of the wave functional that we are trying to explain is that, if we hold $\phi_{\lambda,\text{I}}$ fixed, that $|\Psi_0|^2$ is maximized when

$$\phi_{\lambda,\text{II}} = \frac{1}{\cosh(\pi\lambda)}\phi_{\lambda,\text{I}}, \tag{14.79}$$

where the above equation of course holds for both the real and imaginary parts of $\phi_{\lambda,\text{I}}$ and $\phi_{\lambda,\text{II}}$ as well.

To maximize $|\Psi_0|^2$ as a function of $\text{Re}(\phi_{\lambda,\text{II}})$ while keeping $\text{Re}(\phi_{\lambda,I})$ fixed, we must simply move along a line of constant $\text{Re}(\phi_{\lambda,I})$ until we get to the point which closest to the origin. This is depicted in Figure 27. Geometrically, the point on the line of constant $\text{Re}(\phi_{\lambda,I})$ which is closest to the origin is the point that makes a perpendicular line when connected to the origin. Using a bit of calculus, one can indeed show that $\text{Re}(\phi_{\lambda,\text{II}})$ will indeed be $1/\cosh(\pi\lambda)$ times $\text{Re}(\phi_{\lambda,\text{I}})$, which is indeed what we wanted to show! [8]

To get a feel for why the value of $\text{Re}(\phi_{\lambda,\text{II}})$ which maximizes $|\Psi_0|^2$ will be nearly equal to $\text{Re}(\phi_{\lambda,\text{I}})$ for small λ and nearly equal to 0 for big λ, we shall simply draw the above picture for small λ and big λ, respectively in Figure 28. Hopefully this will give you some visual intuition for

[8] More specifically, if one uses equations (14.77) and (14.78), then one can show that by minimizing the quantity $\text{Re}(\phi_{>,\lambda})^2 + \text{Re}(\phi_{>,-\lambda})^2$ with respect to $\text{Re}(\phi_{\lambda,\text{II}})$ while keeping $\text{Re}(\phi_{\lambda,\text{I}})$ fixed, one recovers exactly (14.79).

why the modes with small λ are heavily entangled between right and left while the modes with large λ are not very entangled.

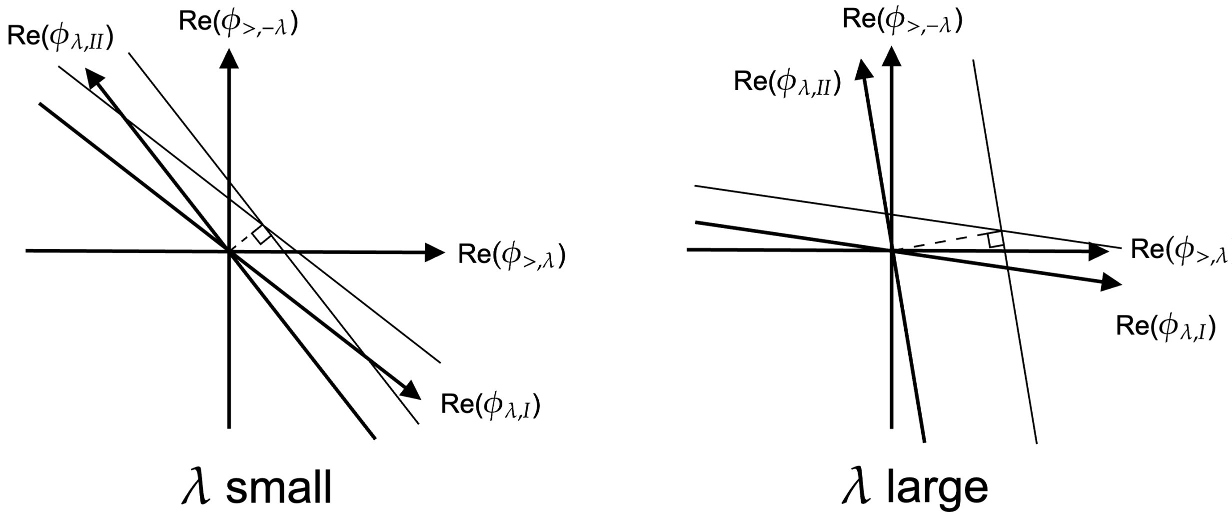

Figure 28: Here we show how the axes look when λ is small and large, respectively. Note that the slopes of the Re($\phi_{\lambda,\mathrm{I}}$) and Re($\phi_{\lambda,\mathrm{II}}$) axes are $-e^{-\pi\lambda}$ and $-e^{\pi\lambda}$, respectively. Carrying out our procedure for finding the value of $\phi_{\lambda,\mathrm{II}}$ which maximises $|\Psi_0|^2$ given a fixed $\phi_{\lambda,\mathrm{I}}$, we can see geometrically that said value of $\phi_{\lambda,\mathrm{II}}$ is very close to $\phi_{\lambda,\mathrm{I}}$ when λ is small and very close to 0 when λ is large.

14.3.6 Bringing it all together

Now that we are done explaining each individual component, let us review the (admittedly long) chain of logic which allows to explain why the ground state wave functional looks thermal to accelerated observer.

1. The natural modes which appear to have constant frequency to accelerating observer are Rindler modes, which for massless fields in 1+1d on a fixed $t = 0$ time slice are eigenfunctions of the scaling operator $x \to e^{\theta}x$. Such global Rindler modes on a time slice, denoted $\Phi_{>,\lambda}(x)$ and $\Phi_{<,\lambda}(x)$ can be defined as integrals of plane waves made from exclusively wave numbers with $k > 0$ and $k < 0$, respectively. These global Rindler modes are averages over all possible scalings of said plane waves multiplied by a λ-dependent phase to make the whole function an eigenfunction under scaling.

2. These global Rindler modes $\Phi_{>,\lambda}(x)$ ($\Phi_{<,\lambda}(x)$) have an amplitude which is larger by a factor of $e^{\pi\lambda}$ (smaller by a factor of $e^{-\pi\lambda}$) for $x > 0$ than $x < 0$. The reason for this is apparent if you look closely as the phases which are being averaged over, as constructive interference of the phases occurs for half of the values of x while destructive interference occurs for the other half, as depicted in Figure 19.

3. Due to the fact that the ground state $\Psi_0[\phi]$ is real (all energy eigenstates can be chosen to have real wave functions) and boost invariant, one can argue that is must be essentially

of the form $\Psi_0[\phi] \sim \exp\left(-\int d\lambda \phi_{>,\lambda}\phi_{<,-\lambda}\right)$ where $\phi^*_{>,\lambda} = \phi_{<,-\lambda}$. The reason that it must be given by an integral over all λ, without any mixing between modes of different λ is because the wave function must be boost invariant and cannot pick up interfering phases between oscillators of different Rindler frequencies under boosts. Switching to real variables, this means that the ground state wave functional can be written as $\Psi_0[\phi] \sim \exp\left(-\int d\lambda (\text{Re}(\phi_{>,\lambda})^2 + \text{Im}(\phi_{>,\lambda})^2)\right)$, where we note that the real and imaginary parts of the global Rindler mode coefficients are independent of each other, i.e. their probability distributions are uncorrelated.

4. We can also deduce that $\text{Re}(\phi_{>,\lambda})$ and $\text{Im}(\phi_{>,\lambda})$ have uncorrelated probability distributions by noting that sine waves and cosine waves are uncorrelated in Ψ_0, and then noting that $\text{Re}(\phi_{>,\lambda})$ and $\text{Im}(\phi_{>,\lambda})$ are just linear combinations of translated sine waves and cosine waves. (See figures 24 and 25.)

5. Once we know that the real and imaginary parts of $\phi_{>,\lambda}$ are uncorrelated, we can inspect the behavior of the real and imaginary parts of the Rindler modes individually. Looking at the real part, we write $\phi_{>,\lambda}$ in terms of the right and left restricted Rindler modes $\phi_{\lambda,\text{I}}$ and $\phi_{\lambda,\text{II}}$. Investigating the way in which the $\text{Re}(\phi_{\lambda,\text{I}})$ and $\text{Re}(\phi_{\lambda,\text{II}})$ axes "skew" on top of the uncorrelated Gaussian probability distribution of $\text{Re}(\phi_{>,\lambda})$ and $\text{Re}(\phi_{>,-\lambda})$, we find that the distributions of $\text{Re}(\phi_{\lambda,\text{I}})$ and $\text{Re}(\phi_{\lambda,\text{II}})$ *are* correlated. In fact, if we hold $\phi_{\lambda,\text{I}}$ fixed, then $|\Psi_0|^2$ is maximized by taking
$$\phi_{\lambda,\text{II}} = \frac{1}{\cosh(\pi\lambda)}\phi_{\lambda,\text{I}}$$
which means that $\phi_{\lambda,\text{I}}$ and $\phi_{\lambda,\text{II}}$ are highly correlated for small λ (low Rindler frequencies) and not very correlated for high λ (large Rindler frequencies). See Figure 28.

6. The fact that we have a high degree of correlation between right and left for small λ but not for large λ means that if we "integrate out" $\phi_{\lambda,\text{II}}$ from the probability distribution, then the variance of the $\phi_{\lambda,\text{I}}$ distribution will be very large for small λ and not very large for large λ. (See Figure 18.) This is the exactly behavior you would expect if the right half of the spacetime has some finite Rindler temperature, where low frequency modes are distributed in highly spread out Gaussians while high frequency modes are essentially frozen out in their oscillator ground state.

7. The Unruh effect exists, QED!

15 Detectors and black hole radiation

While black holes have not been the main focus of this note, our in-depth discussion of particle detectors and the intuition we have built has put us in a good position to answer some questions about what various observers will see when a black hole emits Hawking radiation.

Recall that the Schwarzschild metric is
$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{15.1}$$

At large r, the Schwarzschild time coordinate t behaves just like the Minkowski inertial time. However, near the horizon $r = 2GM$, the coordinate t behaves quite differently. This becomes clear if one looks at the vector field ∂_t on the Penrose diagram for the eternal Schwarzschild back hole, as drawn in 29, we can see that the Schwarzschild time vector field ∂_t acts like a Minkowski boost near the event horizon, with the bifurcation 2-sphere acting as the fixed point.

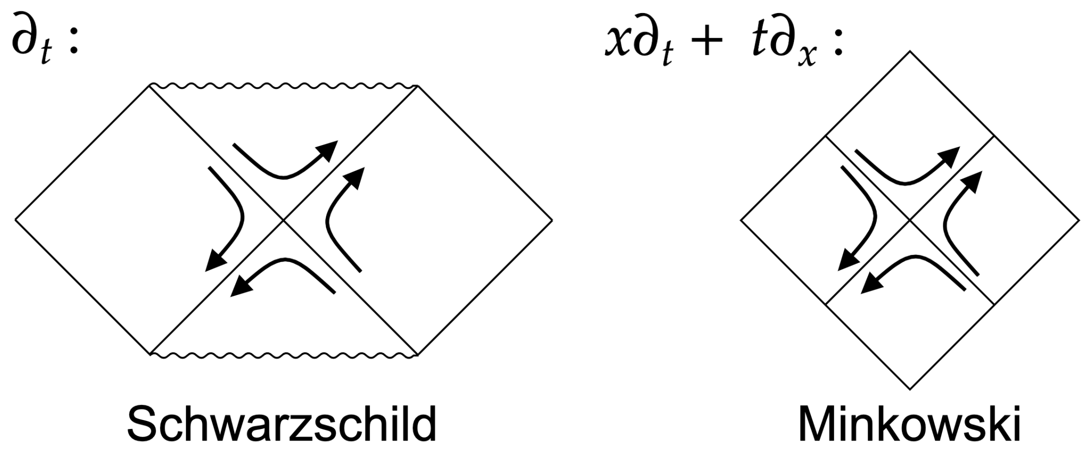

Figure 29: The Killing flow for ∂_t in the Schwarzschild black hole Penrose diagram, versus the boost Killing vector field in the Minkowski space Penrose diagram. Note that Schwarzschild time acts like a boost near the event horizon.

So, crucially, we can see that ∂_t behaves as a *boost* near the horizon, but behaves like *inertial time* far away from the black hole. It is these two facets of ∂_t which have important consequences for Hawking radiation, as we shall now discuss.

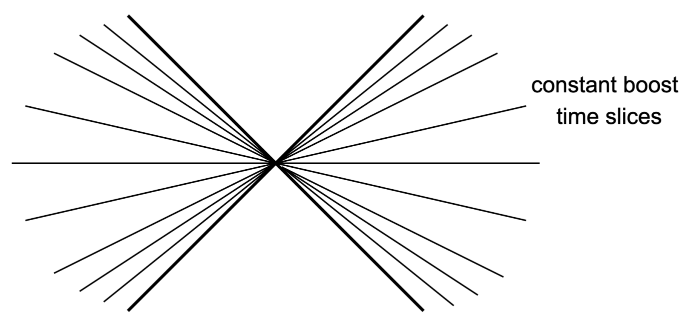

Figure 30: Constant boost-time slices near the horizon fixed point

Because ∂_t is a Killing vector of the Schwarzscild spacetime, it behooves us to define a set of positive and negative frequency modes with respect to it.

$$\partial_t \phi = -i\omega\phi \text{ is positive (negative) frequency if } \omega > 0 \ (\omega < 0) \tag{15.2}$$

Crucially, near the event horizon, these positive frequency modes with respect to ∂_t will look like *Rindler* modes, while far from the event horizon they will look like *Minkowski plane waves*. So, if the modes are thermal on one side the event horizon (just as Rindler modes are thermal on one side of the acceleration horizon in flat space) then as the modes travel outwards and peel away from the horizon, they will emerge in the large r region as "real" inertial particles.

Thus, the fact that ∂_t interpolates between being boost-like-time near the horizon and inertial-like-time far from the black hole means that the outgoing "Rindler modes" at the horizon are converted into "real particles" far away from the black hole! This is exactly the Hawking effect.

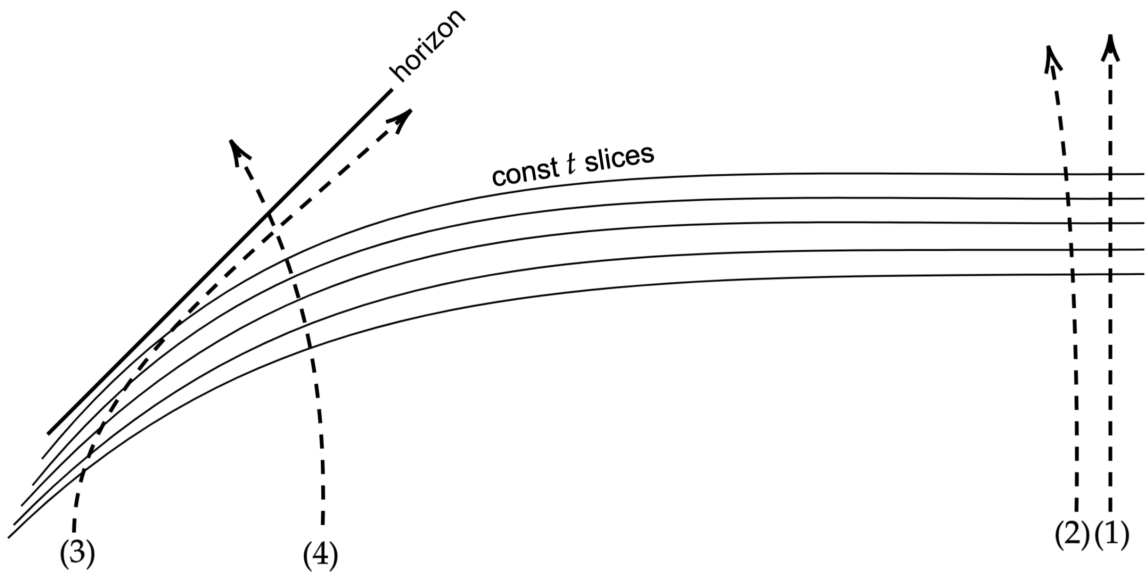

Figure 31: Four different observers in Schwarzschild spacetime, who each observe the outgoing Hawking modes differently. Constant t slices are also drawn, which look like boost time slices near the horizon and inertial time slices far away from the horizon.

Let us now describe what different observers see when a black hole radiates. In particular, there are four possible observers I am interested in discussing:

1. an observer who stays at a constant radius away from the black hole at large r
2. a free-falling observer at large r
3. an observer who stays at a constant radius away from the black hole near the horizon
4. a free-falling observer near the horizon.

The worldlines of these observers are drawn in Figure 31.

We begin with observer 1, who stays at a constant radius far from the black hole. As we have just discussed, they see an outward flux of real Hawking particles (positive frequency with respect to the inertial time ∂_t) leaving the black hole. Because it takes essentially no acceleration to maintain a constant distance from a far away black hole they will see essentially no Unruh radiation due to their acceleration.

Observer 2 is falling very slowly into the black hole. Because observer 2's path and proper time coordinate is almost exactly the same as observer 1's, observer 2 will see almost the exact same thing as observer 1.

I would now like to address a popular misconception about Hawking radiation. You may have heard some version of the following logic:

> Using the equivalence principle, we know that Hawking radiation must exist because we know Unruh radiation exists. In order to maintain a constant distance from a black hole, you must have some acceleration and will therefore see Unruh radiation. But this Unruh radiation is exactly the Hawking radiation! As a corollary, if you are in a free-falling frame, you won't see any Hawking radiation.
>
> —Someone who is wrong

This is completely incorrect. You cannot "derive" the existence of Hawking radiation using the Unruh effect and the equivalence principle. The biggest problem with this logic is that a free-falling observer far from the black hole absolutely *does* see an outward flux of Hawking particles leaving the black hole, because the free-falling observer's notion of time is almost exactly the same as the constant-radius-observer's notion of time at large r. As another way to see why this logic must be wrong, the amount of acceleration necessary to maintain a constant radius shrinks to 0 as $r \to \infty$, while the temperature of the Hawking radiation stays constant as $r \to \infty$. Because the Unruh temperature is $T = a/2\pi$, we can deduce that Hawking radiation cannot simply be the Unruh radiation of a constant-r observer in disguise.

More importantly, the equivalence principle cannot properly be invoked in this scenario for a simple conceptual reason. When discussing what different observers "see," we must always specify which *quantum state* the quantum field is in. For instance, an inertial observer in Minkowski space will see *very* different things if the quantum field is in the Minkowski vacuum $|0_M\rangle$ versus if it is in the Rindler vacuum $|0_R\rangle$. Einstein's equivalence principle (which was formulated before people knew about quantum field theory) makes no reference to the particular state of the quantum field, and thus is not sufficient to make any statement about Hawking radiation.

I think a better way to think of it is as follows: a black hole is an object that converts Rindler modes near the horizon (which you would have to have a large acceleration to detect) to real inertial particles far away from the horizon.

Nontheless, if someone wants to say that Hawking radiation "is" Unruh radiation (whatever that means), I wouldn't necessarily say they are *wrong*, I would just say that it is a very vague statement which needs to be fleshed out to be meaningful, and that the connection between Hawking radiation and Unruh radiation is a bit subtle.

Let us now describe what observer 3 sees, who stays at a constant radius very close to the horizon. This observer is analogous to an accelerating Rindler observer, and their proper time coincides with the boost-like time ∂_t near the horizon. Therefore, because the Hawking modes are essentially outgoing Rindler modes near the horizon, observer 3 will see the Hawking modes possessing a very high energy per particle, due to the high acceleration necessary to remain close to the horizon. In other words, observer 3 will see the Hawking modes as extremely blue-shifted and energetic, with the temperature being extremely hot.

Observer 4, who free-falls through the horizon, has the most complicated perspective. Their notion of time does not align at all with the Schwarzschild time ∂_t. Due to the curvature of the spacetime, they will see some radiation present due to the curvature of spacetime, but this raiation will be extremely low energy, with wavelength on the order of the Schwarzschild radius $\sim 2GM$. Providing an exact mathematical account of what observer 4 sees is tricky, because they fall into the singularity after a finite proper time. Usually, we discuss detectors which are kept "on" for a very long time, and have a long time to reach thermal equilibrium. Because observer 4 doesn't have a long enough time to reach thermal equilibrium, it is unclear if we can assign a meaningful temperature to their observations. However, it is certainly safe to say that observer 4 certainly *does not* see the extremely high temperature highly blueshifted Rindler-like modes that observer 3 sees. Only an accelerating observer whose proper time aligns with the boost time can see those modes.

Having finished our discussion of what the different observers see, I now want to address the question of how the quantum fields around a realistic black hole (formed by stellar collapse) settle into the quantum field state where the modes are essentially Rindler modes near the horizon.

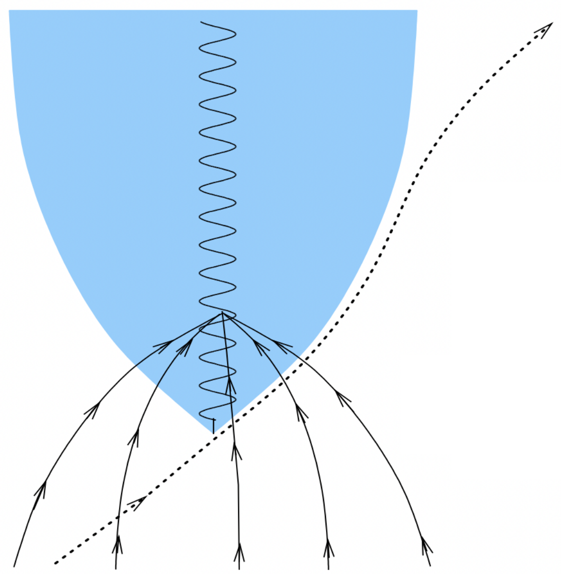

Figure 32: Image taken from [15]. A high-frequency Rindler wave packet is nearly captured by the forming black hole. It then slowly peels off of the horizon until it escapes and is emitted as Hawking radiation. Because the original Rindler mode was entangled with a nearby partner mode which does fall into the black hole, the outgoing mode is entangled with a mode which eventually falls into the singularity and is destroyed.

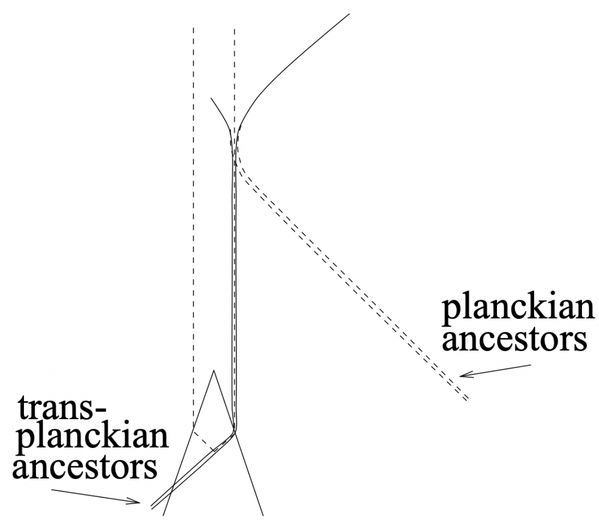

Figure 33: Image taken from [16]. If spacetime is a continuum, the eventual outgoing Hawking modes start out as "transplanckian ancestors" (extremely short wavelength Rindler modes) which are almost captured by the black hole at the time of formation. If spacetime is a lattice planck length spacing, then the Hawking modes start out as ingoing "planckian ancestors," one of which will enter the black hole and the other of which will bounce off of the horizon.

In Hawking's original paper [17], he showed that if a quantum field's state is initially in the Minkowski vacuum before a black hole forms via stellar collapse, the quantum field state near the horizon will contain entangled "Hawking pairs" of modes which are just inside/outside the horizon. As the outgoing modes slowly peel away from the horizon and get redshifted, they will be emitted as a thermal gas of particles with a blackbody spectrum.

To show this, he considered the evolution of a very high-frequency Rindler wave packet which is nearly captured by the black hole as it forms, as is shown in Figure 32. What's more, the original Rindler mode was entangled with a partner mode, just as Rindler modes always are in flat space.[9]The partner mode is then captured by the horizon and falls into the black hole. It should be noted that the original pair of Rindler modes is entangled in flat space before the black hole forms—the black hole is just the agent that captures one of them and emits the other one after considerable redshifting. In this way, we can once again understand black holes as objects which convert Unruh radiation into actual radiation.

However, there is something about this story which unsettles people: the Rindler modes which are nearly captured by the black hole and then slowly peel away from the horizon start out with an absurdly small wavelength, much smaller than the planck length. While these modes are eventually reshifted into modes with long wavelengths, people fear that the fact that they originally had such a short wavelength will make this whole analysis invalid due to small distance quantum gravity effects. This is called the "transplanckian problem" (although nowawdays it is not considered to be much of a problem).

One possible solution to the transplanckian problem is described nicely in the wonderful note [16]. If one creates a model quantum field theory on a discrete lattice with planck length spacing, what one finds is that the modes which eventually come out as Hawking radiation actually started out as *ingoing* modes, which then "bounce" off the horizon. This is drawn in Figure 33. The cause of the bounce is the finite length of the lattice spacing, for reasons not dissimilar to how a wheel which spins very quickly may appear to spin backwards when filmed by a camera with a fixed framerate.

However, whether or not the ancestors are "transplanckian" or "planckian" does not make much a difference to the final Hawking radiation. In fact, in a sense, all that really matters is that the quantum field state at the horizon is "smooth" and approximates the ground state as seen by a free-falling observer. All you *really* need to derive the Hawking effect is to assume that the quantum field state at the horizon is "nice," in the same way that the ground state is "nice" according to an inertial observer in Minkowski space. A detailed account of the ancestor modes is not really necessary, although it is interesting to think about.

[9]Let me remind you what we mean when we say two modes are "entangled." If we have a complete basis of positive frequency modes (for some notion of 'positive frequency') then states in the Hilbert space can be labelled by an occupation number for each mode. This occupation number corresponds to how many particles there are in that mode. For two modes to be "entangled" in a state means that that the occupation number of one mode is correlated with the occupation number of the other mode. In our case, if the occupation number of the outgoing mode is n, then the occupation number of the infalling partner mode is also n.

A Intuitive understanding of entanglement between right and left from wave functional

In this appendix, we present a way to understand "why" the quantum fields in the right and left halves of spacetime are entangled in the ground state. This is not quite as detailed as the understanding presented in section 14 and has thus been relegated to the appendix.

We start with the expression for the ground state wave functional for a massless real scalar field in 1+1d:

$$\Psi_0[\phi] \propto \exp\left(-\frac{1}{\hbar}\int_0^\infty dk\,|k|\phi_k\phi_{-k}\right).$$

How does this wave functional behaves qualitatively? It takes its largest value when $\phi(x) = 0$. The larger the Fourier components of the classical field, the smaller Ψ_0 is. Therefore the wave functional outputs very tiny number for classical fields that are far from 0. Furthermore, because of the $|k|$ term, the high frequency Fourier components are penalized more heavily that the low frequency Fourier components. Therefore, the wave functional Ψ_0 is very small for big and jittery classical fields, and very large for small and gradually varying classical fields.

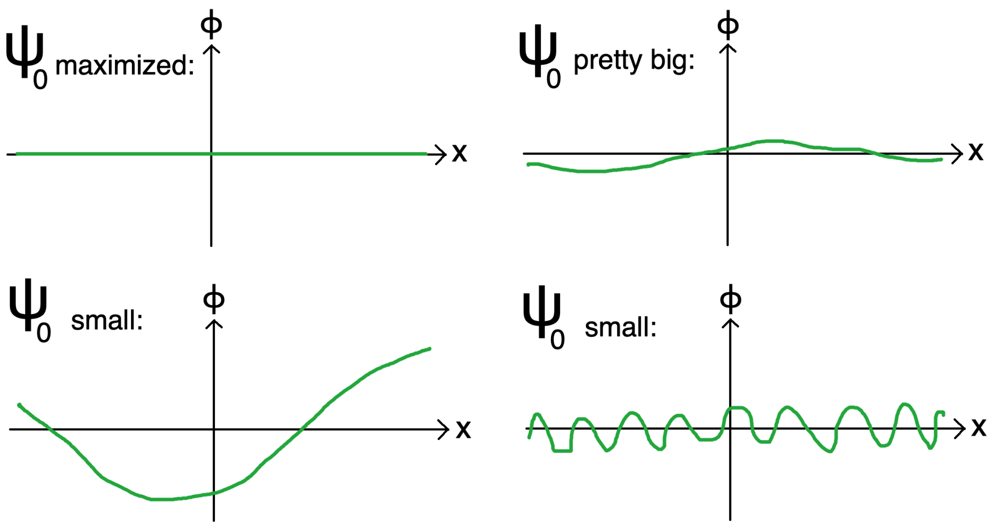

Figure 34: Some sample classical field configurations and the relative size of Ψ_0 when evaluated at each one. The upper-left field maximizes Ψ_0 because it is 0. The upper-right field is pretty close to 0, so Ψ_0 is still pretty big. The lower-left field makes Ψ_0 small because it contains large field values. The lower-right field makes Ψ_0 because its frequency $|k|$ is large even though the Fourier-coefficient ϕ_k is not that large.

Now, the statement that the right and left halves of spacetime are "entangled" or "correlated" simply means that if we specify what $\phi(x)$ is in the $x > 0$ region, then are some ways to complete $\phi(x)$ in the $x < 0$ region which are more probable than others.

For instance, a configuration of $\phi(x)$ where $\phi(x)$ is some non-zero function for $x > 0$ and then instantly drops to $\phi(x) = 0$ for $x < 0$ is very unlikely. Because $\phi(x)$ has a "sharp drop" at $x = 0$, such a field contains many very high-frequency Fourier modes and as such is heavily penalized. A much more likely scenario is to have $\phi(x)$ continue smoothly into the $x < 0$ region and slowly go to zero over some distance. This is the way to minimize the size of the Fourier modes while still retaining the given value of the function for $x > 0$. This is depicted in Figure 35.

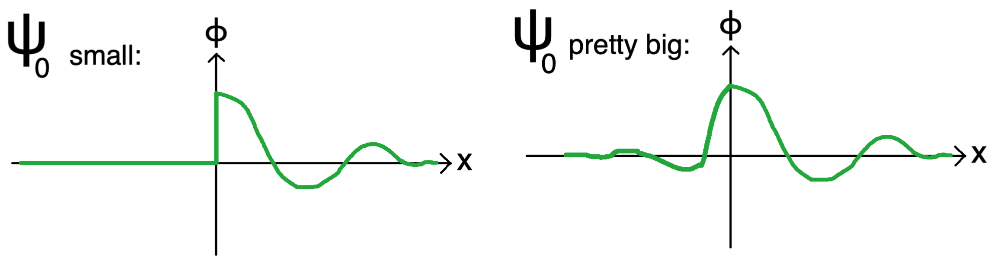

Figure 35: Here we have two field configurations with a certain fixed profile of $\phi(x)$ for $x > 0$ and two possible completions for $x < 0$. A sharp drop-off at $x = 0$ renders Ψ_0 small due to the presence of high frequency Fourier modes, while a smoother continuation renders Ψ_0 much larger.

B Two oscillator toy model

In this appendix we present a toy model for Unruh radiation using two quantum harmonic oscillators.

Say we have two independent oscillator algebras

$$[\hat{a}_{\rm I}, \hat{a}_{\rm I}^\dagger] = 1, \qquad [\hat{a}_{\rm II}, \hat{a}_{\rm II}^\dagger] = 1 \tag{B.1}$$

with all other commutators being zero. We define the joint ground state as

$$\hat{a}_{\rm I} \left|\Omega\right> = \hat{a}_{\rm II} \left|\Omega\right> = 0. \tag{B.2}$$

Let us now define the "rotated" creation and annihilation operators $\hat{b}_{\rm I}$, $\hat{b}_{\rm II}$, $\hat{b}_{\rm I}^\dagger$, $\hat{b}_{\rm II}^\dagger$ by

$$\hat{b}_{\rm I} = \cosh\varphi\, \hat{a}_{\rm I} + \sinh\varphi\, \hat{a}_{\rm II}^\dagger \tag{B.3}$$
$$\hat{b}_{\rm I}^\dagger = \cosh\varphi\, \hat{a}_{\rm I}^\dagger + \sinh\varphi\, \hat{a}_{\rm II} \tag{B.4}$$
$$\hat{b}_{\rm II} = \sinh\varphi\, \hat{a}_{\rm I}^\dagger + \cosh\varphi\, \hat{a}_{\rm II} \tag{B.5}$$
$$\hat{b}_{\rm II}^\dagger = \sinh\varphi\, \hat{a}_{\rm I} + \cosh\varphi\, \hat{a}_{\rm II}^\dagger \tag{B.6}$$

with the inverse relations

$$\hat{a}_{\rm I} = \cosh\varphi\, \hat{b}_{\rm I} - \sinh\varphi\, \hat{b}_{\rm II}^\dagger \tag{B.7}$$
$$\hat{a}_{\rm I}^\dagger = \cosh\varphi\, \hat{b}_{\rm I}^\dagger - \sinh\varphi\, \hat{b}_{\rm II} \tag{B.8}$$
$$\hat{a}_{\rm II} = -\sinh\varphi\, \hat{b}_{\rm I}^\dagger + \cosh\varphi\, \hat{b}_{\rm I} \tag{B.9}$$
$$\hat{a}_{\rm II}^\dagger = -\sinh\varphi\, \hat{b}_{\rm II} + \cosh\varphi\, \hat{b}_{\rm II}^\dagger. \tag{B.10}$$

These new creations and annihilation operators satisfy the commutation relations

$$[\hat{b}_{\rm I}, \hat{b}_{\rm I}^\dagger] = 1, \qquad [\hat{b}_{\rm II}, \hat{b}_{\rm II}^\dagger] = 1. \tag{B.11}$$

Eq. (B.2) then becomes

$$\hat{b}_{\rm I} |\Omega\rangle = \tanh\varphi\, \hat{b}_{\rm II}^\dagger |\Omega\rangle\,, \qquad \hat{b}_{\rm II} |\Omega\rangle = \tanh\varphi\, \hat{b}_{\rm I}^\dagger |\Omega\rangle\,. \tag{B.12}$$

If we define the b vacuum and excited states by

$$\hat{b}_{\rm I} |0,0\rangle = \hat{b}_{\rm II} |0,0\rangle = 0, \qquad |m,n\rangle = \frac{(\hat{b}_{\rm I}^\dagger)^m}{\sqrt{m!}} \frac{(\hat{b}_{\rm II}^\dagger)^n}{\sqrt{n!}} |0,0\rangle\,, \tag{B.13}$$

then Eq. (B.12) can be solved to yield

$$|\Omega\rangle = \exp\Big(\tanh\varphi\, \hat{b}_{\rm I}^\dagger\, \hat{b}_{\rm II}^\dagger\Big) |0,0\rangle \tag{B.14}$$

$$= \sum_{n=0}^{\infty} (\tanh\varphi)^n |n,n\rangle\,. \tag{B.15}$$

If we make the substitution

$$\cosh\varphi = \frac{1}{\sqrt{1-e^{-\beta\omega}}}, \qquad \sinh\varphi = \frac{e^{-\beta\omega/2}}{\sqrt{1-e^{-\beta\omega}}}, \qquad \tanh\varphi = e^{-\beta\omega/2}, \tag{B.16}$$

then this becomes the more familiar expression

$$|\Omega\rangle = \sum_{n=0}^{\infty} e^{-n\beta\omega/2} |n,n\rangle \tag{B.17}$$

reminiscent of how the Minkowski vacuum is a thermofield double entangled state built on top of the Rindler vacuum. In this analogy, the operators $\hat{a}$ correspond to the global positive frequency modes which annihilate the Minkowski vacuum while the operators $\hat{b}$ correspond to the Rindler modes restricted onto either region I and II. See, for instance, Eq. (7.16).

Extending the analogy a bit further, the "boost" Hamiltonian $\hat{H}$ should be

$$\hat{H} = \omega \hat{b}_{\rm I}^\dagger \hat{b}_{\rm I} - \omega \hat{b}_{\rm II}^\dagger \hat{b}_{\rm II} \tag{B.18}$$

such that $\hat{H}|\Omega\rangle = 0$. Notice that one of the oscillators has positive energies while the other one has negative energies. This is reminiscent of how the boost vector field points towards the future in one Rindler wedge and towards the past in the other Rindler wedge.

C Sinc function to delta function

We are interested in the following integral as $T \to \infty$:

$$\int_0^T dt\, e^{i\omega_0 t} e^{-i\omega t} = \frac{e^{i(\omega_0-\omega)T} - 1}{i(\omega_0-\omega)} \tag{C.1}$$

$$= 2e^{-i(\omega_0-\omega)T/2} \frac{\sin((\omega_0-\omega)T/2)}{(\omega_0-\omega)}. \tag{C.2}$$

The modulus squared of the integral is

$$\left|\int_0^T dt\ e^{i\omega_0 t}e^{-i\omega t}\right|^2 = 4\frac{\sin^2((\omega_0-\omega)T/2)}{(\omega_0-\omega)^2}. \tag{C.3}$$

Note the identity

$$\lim_{T\to\infty}\frac{4\sin^2(\alpha T/2)}{\alpha^2 T} = 2\pi\delta(\alpha) \tag{C.4}$$

which can be proven by integrating the left hand side against $\int d\alpha$ and seeing it is 2π. (Notice that the above equation is approximately true as long as $\alpha T \gg 1$.) This means that

$$\lim_{T\to\infty}\left|\int_0^T dt\ e^{i\omega_0 t}e^{-i\omega t}\right|^2 = 2\pi T\delta(\omega_0-\omega). \tag{C.5}$$

D Fermi's Golden Rule

D.1 Derivation

In this appendix we restore factors of $\hbar$. Consider the Hamiltonian

$$H(t) = H_0 + W(t) \tag{D.1}$$

where we assume $W(t)$ is small, in the sense that it involves a small coupling constant. Say that the free Hamiltonian H_0 has eigenstates denoted by $|n\rangle$ with eigenvalues E_n:

$$H_0\,|n\rangle = E_n\,|n\rangle\,. \tag{D.2}$$

Now, let's say that a state evolves in time under the full Hamiltonian $H(t)$ as

$$|\psi(t)\rangle = \sum_n c_n(t)\,|n\rangle\,e^{-iE_n t/\hbar}. \tag{D.3}$$

If we insert the above decomposition into the equation

$$\langle k|\,i\hbar\partial_t\,|\psi(t)\rangle = \langle k|\,(H_0+W(t))\,|\psi(t)\rangle \tag{D.4}$$

we get

$$\frac{d}{dt}c_k(t) = -\frac{i}{\hbar}\sum_n e^{-i(E_n-E_k)t/\hbar}c_n(t)\,\langle k|\,W(t)\,|n\rangle\,. \tag{D.5}$$

This can be integrated to get

$$c_k(t_f) = c_k(t_i) - \frac{i}{\hbar}\int_{t_i}^{t_f} dt\sum_n e^{-i(E_n-E_k)t/\hbar}c_n(t)\,\langle k|\,W(t)\,|n\rangle \tag{D.6}$$

Let's now consider what happens to an initial state $|i\rangle$. If we solve this equation pertubatively, plugging in the 0^{th} order solution to (D.6), which is just $c_n(t) = \delta_{ni}$, we get

$$\begin{aligned} c_k(t_f) &= \delta_{ki} - \frac{i}{\hbar}\int_{t_i}^{t_f} dt\ e^{-i(E_i-E_k)t/\hbar}\,\langle k|\,W(t)\,|i\rangle \\ &= \delta_{ki} - \frac{i}{\hbar}\int_{t_i}^{t_f} dt\ \langle k|\,e^{iH_0 t/\hbar}W(t)e^{-iH_0 t/\hbar}\,|i\rangle\,. \end{aligned} \tag{D.7}$$

Strictly speaking, this approximation is only good for short times, before the $c_k(t)$'s for $k \neq i$ grow appreciably from their initial value of 0. So, if we denote the magnitude of the interaction term as $\langle W \rangle$, and we leave the detector on for a length of time T, then this approximation requires $\langle W \rangle T/\hbar \ll 1$.

Now, let's specialize to a time dependent interaction term of the form

$$W(t) = W e^{i\omega_0 t} + W^\dagger e^{-i\omega_0 t}. \tag{D.8}$$

where

$$\omega_0 > 0. \tag{D.9}$$

Let's compute the probability to go from state $|i\rangle$ to state $|f\rangle$, which are both eigenstates of the free Hamiltonian H_0 with eigenvalues E_i and E_f. Let us also assume for simplicity that

$$E_i > E_f. \tag{D.10}$$

To the first order in perturbation theory, Fermi's golden rule tells us that this probability is

$$P_{i\to f} = \frac{1}{\hbar^2} \left| \int_{t_i}^{t_f} dt \; e^{-i(E_i - E_f)t/\hbar} \langle f | \, W(t) \, | i \rangle \right|^2 . \tag{D.11}$$

Notice that in the integral in the above expression can be expanded as

$$\begin{aligned} &\int_{t_i}^{t_f} dt \; \left(e^{-i(E_i-E_f)t/\hbar} e^{i\omega_0 t} \langle f | \, W \, | i \rangle + \cancel{e^{-i(E_i-E_f)t/\hbar} e^{-i\omega_0 t} \langle f | \, W^\dagger \, | i \rangle} \right) \\ &= -i\hbar \langle f | \, W \, | i \rangle \; \frac{e^{it(E_f - E_i + \hbar\omega_0)/\hbar}}{E_f - E_i + \hbar\omega_0} \Bigg|_{t=t_i}^{t=t_f} - \cancel{i\hbar \langle f | \, W^\dagger \, | i \rangle \; \frac{e^{it(E_f - E_i - \hbar\omega_0)/\hbar}}{E_f - E_i - \hbar\omega_0}} \Bigg|_{t=t_i}^{t=t_f}. \end{aligned} \tag{D.12}$$

Note that we can throw out the second term due to the "rotating wave approximation." This essentially just means that, because $E_i - E_f > 0$ and $\omega_0 > 0$, the denominator in the second term is always going to be a lot smaller than the denominator in the first term, which dominates. So, with the rotating wave approximation, we now write

$$P_{i\to f} = \frac{1}{\hbar^2} \left| \int_{t_i}^{t_f} dt \; e^{-i(E_i - E_f)t/\hbar} e^{i\omega_0 t} \langle f | \, W \, | i \rangle \right|^2 . \tag{D.13}$$

Using appendix C, this becomes

$$P_{i\to f} = \frac{2\pi T}{\hbar^2} \delta(\omega_0 - (E_i - E_f)/\hbar) \left| \langle f | \, W \, | i \rangle \right|^2 \tag{D.14}$$

where $T = t_f - t_i$. Differentiating by T, the transition rate per unit time is therefore

$$\frac{d}{dT} P_{i\to f} = \frac{2\pi}{\hbar^2} \delta(\omega_0 - (E_i - E_f)/\hbar) \left| \langle f | \, W \, | i \rangle \right|^2 . \tag{D.15}$$

It's worth discussing the regimes of validity for the above formulae. (Notice by the way that we have assumed a continuum of energy levels– if they were discrete then the probabiltiy would grow as T^2 and not T!) We have assumed that both $\langle W \rangle T/\hbar \ll 1$ and

$$T(\omega_0 - (E_i - E_f)/\hbar) \gg 1. \tag{D.16}$$

So in other words, the time T must be much longer than the inverse difference between $\hbar\omega_0$ and the change in energy level of the state (which is the Heisenberg time-energy uncertainty relation), and the coupling $\langle W \rangle$ must be so weak that the quantum state does not change too much during this evolution. These two conditions may seem too restrictive to be useful. However, if $W(t)$ represents the interaction of some measurement process, and there is some constant source of decoherence between the state and the outside environment, then it is the transition rate $dP_{i\to f}/dt$ which is really being measured in between decoherence events, and our assumptions about the relative sizes of our given parameters actually turn out to be quite reasonable.

D.2 Total transition probability

Let us also compute the probability that *any* transition occurs, i.e. for $i \to f$ where f can be any state. If $f = i$, we subtract 1 so as to discount the "no transition" possibility. We use (D.7) to write

$$
\begin{aligned}
P_{\text{tot}} &= \sum_f |c_f(t_f) - \delta_{fi}|^2 && \text{(D.17)} \\
&= \frac{1}{\hbar^2} \int_{t_i}^{t_f} dt' \int_{t_i}^{t_f} dt \sum_f \langle i| \, e^{iH_0 t'/\hbar} W(t') e^{-iH_0 t'/\hbar} \, |f\rangle \, \langle f| \, e^{iH_0 t/\hbar} W(t) e^{-iH_0 t/\hbar} \, |i\rangle && \text{(D.18)} \\
&= \frac{1}{\hbar^2} \int_{t_i}^{t_f} dt' \int_{t_i}^{t_f} dt \; \langle i| \, e^{iH_0 t'/\hbar} W(t') e^{-iH_0 (t'-t)/\hbar} W(t) e^{-iH_0 t/\hbar} \, |i\rangle \, . && \text{(D.19)}
\end{aligned}
$$

E Calculation of two point function

In this appendix we will calculate that the two point function of a massless scalar field in 1+1 dimensions is

$$\langle 0| \, \hat{\phi}(t,x) \hat{\phi}(0,0) \, |0\rangle = -\frac{1}{4\pi} \log\big(t^2 - x^2 - i\epsilon \, \text{sgn}(t)\big). \tag{E.1}$$

We begin the computation by writing down the standard mode decomposition of the field operator

$$\hat{\phi}(t,x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{2|k|} \left(\hat{a}_k e^{-i|k|t + ikx} + \hat{a}_k^\dagger e^{i|k|t - ikx} \right) \tag{E.2}$$

where the creation and annihilation operators are defined via

$$
\begin{aligned}
\hat{a}_k &= \hat{a}(e^{-i|k|t + ikx}) && \text{(E.3)} \\
\hat{a}_k^\dagger &= \hat{a}^\dagger(e^{-i|k|t + ikx}) && \text{(E.4)}
\end{aligned}
$$

and have the commutation relation

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = (e^{-i|k|t + ikx}, e^{-i|k'|t + ik'x})_{\text{KG}} = 4\pi |k| \delta(k - k'). \tag{E.5}$$

We then compute

$$\langle 0|\,\hat{\phi}(t,x)\hat{\phi}(0,0)\,|0\rangle = \int \frac{dk\,dk'}{(2\pi)^2}\frac{1}{4|k||k'|}\langle 0|\,\hat{a}_k e^{-i|k|t+ikx}\hat{a}^\dagger_{k'}\,|0\rangle \tag{E.6}$$

$$= \int_{-\infty}^{\infty}\frac{dk}{4\pi|k|}e^{-i|k|t+ikx} \tag{E.7}$$

$$= \int_0^{\infty}\frac{dk}{4\pi k}\left(e^{-ik(t-x)}+e^{ik(t+x)}\right). \tag{E.8}$$

In order to compute this integral, we use the following trick, differentiating both sides by t:

$$\partial_t\,\langle 0|\,\hat{\phi}(t,x)\hat{\phi}(0,0)\,|0\rangle = -i\int_0^{\infty}\frac{dk}{4\pi}\left(e^{-ik(t-x-i\epsilon)}+e^{-ik(t+x-i\epsilon)}\right) \tag{E.9}$$

$$= -\frac{1}{4\pi}\left(\frac{1}{t-x-i\epsilon}+\frac{1}{t+x-i\epsilon}\right) \tag{E.10}$$

$$= -\frac{1}{4\pi}\frac{2(t-i\epsilon)}{(t-i\epsilon)^2-x^2} \tag{E.11}$$

where the $t \to t - i\epsilon$ proscription was used in the first line so that the integral converged.

Integrating the above equation over t, we then get

$$\langle 0|\,\hat{\phi}(t,x)\hat{\phi}(0,0)\,|0\rangle = -\frac{1}{4\pi}\log\big((t-i\epsilon)^2-x^2\big) \tag{E.12}$$

$$= -\frac{1}{4\pi}\log\big(t^2-x^2-i\epsilon\,\mathrm{sgn}(t)\big) \tag{E.13}$$

as desired.

F Green's functions in 1+1 dimensions

In this appendix, we will review basic properties of the advanced and retarded propagators, as well as the radiative propagator.

The advanced and retarded propagators $G_{\rm adv}$ and $G_{\rm ret}$ both solve the inhomogenous wave equation with delta function source

$$(\partial_t^2-\partial_x^2)G_{\rm adv}(t,x) = \delta(t)\delta(x) \tag{F.1}$$

$$(\partial_t^2-\partial_x^2)G_{\rm ret}(t,x) = \delta(t)\delta(x). \tag{F.2}$$

The difference between the two Green's functions is their boundary conditions. The advanced propagator is defined to vanish in the future while the retarded propagator is defined to vanish in the past.

$$G_{\rm adv}(t,x) = 0 \text{ for } t>0 \tag{F.3}$$

$$G_{\rm ret}(t,x) = 0 \text{ for } t<0. \tag{F.4}$$

Let us now solve for these Green's functions, starting with G_{ret}. This is easier to solve in u and v coordinates, where (F.2) becomes

$$4\partial_u \partial_v G_{\text{ret}} = 2\delta(u)\delta(v) \tag{F.5}$$

allowing us to easily see that

$$G_{\text{ret}} = \frac{1}{2}\Theta(u)\Theta(v) \tag{F.6}$$

$$= \frac{1}{2}\Theta(t-x)\Theta(t+x). \tag{F.7}$$

We can get the advanced Green's function by sending $t \to -t$ in the above equation.

$$G_{\text{adv}} = \frac{1}{2}\Theta(-v)\Theta(-u) \tag{F.8}$$

$$= \frac{1}{2}\Theta(-t-x)\Theta(-t+x). \tag{F.9}$$

The radiative propagator G_{rad} is defined to be the difference of the advanced and retarded propagators.

$$G_{\text{rad}} \equiv G_{\text{adv}} - G_{\text{ret}} \tag{F.10}$$

$$= \frac{1}{2}\Theta(u)\Theta(v) - \frac{1}{2}\Theta(-u)\Theta(-v) \tag{F.11}$$

$$= \frac{1}{4}\text{sgn}(t-x) + \frac{1}{4}\text{sgn}(t+x). \tag{F.12}$$

The radiative propagator satisfies the homogenous wave equation (with no source)

$$(\partial_t^2 - \partial_x^2)G_{\text{rad}} = 0. \tag{F.13}$$

G Hilbert space boost generator $\hat{K}$

In this appendix we will derive the Hilbert space boost generator $\hat{K}$ for a massless free scalar field in 1+1d using Noether's theorem. We will then show that the boost generator "works as it should." We use the convention $\eta^{\mu\nu} = \text{diag}(+,-)$

The classical field Lagrangian is

$$\mathcal{L} = \frac{1}{2}\left((\partial_t\phi)^2 - (\partial_x\phi)^2\right) \tag{G.1}$$

and the boost vector field K is

$$K = t\partial_x + x\partial_t. \tag{G.2}$$

An infinitesimal boost will transform the field as

$$\phi(t,x) \mapsto \phi(t - \varepsilon x, x - \varepsilon t) = \phi(t,x) + \delta\phi(t,x) \tag{G.3}$$

where ε is a tiny constant, and the tiny variation $\delta\phi = -\varepsilon K\phi$ is given by

$$\delta\phi = -\varepsilon(t\partial_x\phi + x\partial_t\phi). \tag{G.4}$$

Let us check to see if this a symmetry of our action, meaning that the above variation only changes $\mathcal{L}$ up to a total derivative. We compute

$$\begin{aligned}
\delta\mathcal{L} &= -\varepsilon\Big((\partial_t\phi)\partial_t(t\partial_x\phi + x\partial_t\phi) - (\partial_x\phi)\partial_x(t\partial_x\phi + x\partial_t\phi)\Big) && \text{(G.5)}\\
&= -\varepsilon\Big(\cancel{(\partial_t\phi)(\partial_x\phi)} + t(\partial_t\phi)(\partial_t\partial_x\phi) + x(\partial_t\phi)(\partial_t^2\phi) && \text{(G.6)}\\
&\qquad - t(\partial_x\phi)(\partial_x^2\phi) - \cancel{(\partial_x\phi)(\partial_t\phi)} - x(\partial_t\phi)(\partial_x\partial_t\phi)\Big) && \text{(G.7)}\\
&= -\varepsilon\frac{1}{2}\Big(\partial_x(t(\partial_t\phi)^2 - t(\partial_x\phi)^2) + \partial_t(x(\partial_t\phi)^2 - x(\partial_x\phi)^2)\Big) && \text{(G.8)}\\
&= -\varepsilon\partial_\mu C^\mu && \text{(G.9)}
\end{aligned}$$

where

$$C^\mu = (x\mathcal{L}, t\mathcal{L}). \tag{G.10}$$

Because $\mathcal{L}$ changes by a total derivative, we have confirmed that boosts are indeed a symmetry of our Lagrangian.

Now we shall use the Noether trick in order to solve for the conserved current associated with boost symmetry. In order to do this, we shall consider the variation $\delta\phi$ where we turn ε into a tiny spacetime function $\varepsilon = \varepsilon(x)$ with compact support:

$$\delta\phi = -\varepsilon(x)(t\partial_x\phi + x\partial_t\phi). \tag{G.11}$$

This variation, like any variation, will leave the action stationary $\delta S = 0$ around solutions to the equations of motion. We shall now expand out the condition $\delta S = 0$. Because we conveniently have already computed the variation of the Lagrangian when ε was a constant, the only new terms we will pick up will be those of the form $\partial_\mu\varepsilon$ where a derivative hits ε:

$$\begin{aligned}
0 &= \delta S && \text{(G.12)}\\
&= \int dt\, dx\Big(- \varepsilon\partial_\mu C^\mu - ((\partial_t\phi)(\partial_t\varepsilon) - (\partial_x\phi)(\partial_x\varepsilon))(t\partial_x\phi + x\partial_t\phi)\Big) && \text{(G.13)}\\
&= \int dt\, dx\varepsilon\ \Big(- \partial_\mu C^\mu + \partial_t\big((\partial_t\phi)(t\partial_x\phi + x\partial_t\phi)\big) - \partial_x\big((\partial_t\phi)(t\partial_x\phi + x\partial_t\phi)\big)\Big) && \text{(G.14)}\\
&= \int dt\, dx\ \varepsilon(\partial_\mu J^\mu) && \text{(G.15)}
\end{aligned}$$

where

$$J^\mu = -C^\mu + \eta^{\mu\nu}(\partial_\nu\phi)(t\partial_x\phi + x\partial_t\phi). \tag{G.16}$$

We then define the classical "boost energy" K_{cl} as the integral of J^t over a $t = 0$ slice via

$$
\begin{aligned}
K_{\mathrm{cl}} &= \int dx\, J^t && \text{(G.17)}\\
&= \int dx \Big(-x\mathcal{L} + x(\partial_t \phi)^2 \Big) && \text{(G.18)}\\
&= \int dx\, x \frac{1}{2}\big((\partial_t \phi)^2 + (\partial_x \phi)^2\big) && \text{(G.19)}\\
&= \int dx\, x T^{00} && \text{(G.20)}
\end{aligned}
$$

where T^{00} is the energy density of the field.

Now that we have an expression for the classical boost energy, we simply put hats on everything to get the quantum operator.

$$
\hat{K} = \frac{1}{2}\int dx\, x(\hat{\pi}^2 + (\partial_x \hat{\phi})^2) \tag{G.21}
$$

Notice that we have the $t = 0$ commutation relations

$$
\begin{aligned}
[\hat{K}, \hat{\phi}(x)] &= -ix\hat{\pi}(x) & [\hat{K}, \partial_t \hat{\phi}(x)] &= [\hat{K}, \hat{\pi}(x)]\\
&= -ix\partial_t \hat{\phi}(x) & &= -i\partial_x (x\partial_x \hat{\phi}(x))\\
&= -iK\hat{\phi}(x) & &= -i\partial_x \hat{\phi}(x) - ix\partial_x^2 \hat{\phi}(x)\\
& & &= -i\partial_x \hat{\phi}(x) - ix\partial_t \hat{\pi}(x)\\
& & &= -i\partial_t (t\partial_x \hat{\phi}(x) + x\partial_t \hat{\phi}(x))\\
& & &= -i\partial_t (K\hat{\phi}(x))
\end{aligned} \tag{G.22}
$$

We now use a nice property of the vector field K: If we have two functions f and g that satisfy the wave equation, and define the tiny boost variations δf and δg as

$$
\delta f = Kf = (t\partial_x f + x\partial_t f), \qquad \delta g = Kg = (t\partial_x g + x\partial_t g) \tag{G.23}
$$

then one can prove using a somewhat laborious series of steps (including many integration by parts) that

$$
\int dx(\delta f^* \partial_t g - g\partial_t \delta f^*) + \int dx(f^* \partial_t \delta g - \delta g \partial_t f^*) = 0 \tag{G.24}
$$

but I shall spare you the step-by-step proof. The above equation can be expressed quite nicely as

$$
(Kf, g)_{\mathrm{KG}} + (f, Kg)_{\mathrm{KG}}. \tag{G.25}
$$

Using our $t = 0$ commutation relations

$$
[\hat{K}, \hat{\phi}(x)] = -iK\hat{\phi}(x), \qquad [\hat{K}, \partial_t \hat{\phi}(x)] = -i\partial_t (K\hat{\phi}(x)) \tag{G.26}
$$

we can immediately compute the commutator $[\hat{K}, \hat{a}^\dagger(f)]$ via

$$[\hat{K}, \hat{a}^\dagger(f)] = [\hat{K}, -(f,^*, \hat{\phi})_{\mathrm{KG}}] \tag{G.27}$$
$$= -(f^*, [\hat{K}, \hat{\phi}])_{\mathrm{KG}} \tag{G.28}$$
$$= -(f^*, -iK\hat{\phi})_{\mathrm{KG}} \tag{G.29}$$
$$= -i(Kf^*, \hat{\phi})_{\mathrm{KG}} \tag{G.30}$$
$$= i\, \hat{a}^\dagger(Kf). \tag{G.31}$$

This is a very important property, as it shows that $\hat{K}$ will act on single particle states $\hat{a}(f)\,|0\rangle$ "as it should."

Because the vacuum state is Lorentz invariant, we have

$$\hat{K}\,|0\rangle = 0. \tag{G.32}$$

Using the above property, we can compute the action of $\hat{K}$ on the single particle state $\hat{a}^\dagger(f)\,|0\rangle$ via

$$\hat{K}\hat{a}^\dagger(f)\,|0\rangle = [\hat{K}, \hat{a}^\dagger(f)]\,|0\rangle \tag{G.33}$$
$$= i\hat{a}^\dagger(Kf)\,|0\rangle\,. \tag{G.34}$$

This means that if the function f is a boost eigenfunction, i.e. if it satisfies

$$Kf = -i\lambda f \tag{G.35}$$

for some constant λ, then we have

$$\hat{K}\hat{a}^\dagger(f) = \lambda\, \hat{a}^\dagger(f)\,|0\rangle \tag{G.36}$$

and so $\hat{K}$ really does behave "as it should."

We can also define the operators $\hat{K}_{\mathrm{I}}$ and $\hat{K}_{\mathrm{II}}$, which is the boost operator restricted to region I ($x > 0$) and region II ($x < 0$)

$$\hat{K}_{\mathrm{I}} = \frac{1}{2}\int_0^\infty dx\; x\big(\hat{\pi}^2 + (\partial_x\hat{\phi}^2)\big), \qquad \hat{K}_{\mathrm{II}} = \frac{1}{2}\int_{-\infty}^0 dx\; x\big(\hat{\pi}^2 + (\partial_x\hat{\phi}^2)\big) \tag{G.37}$$

with

$$\hat{K} = \hat{K}_{\mathrm{I}} + \hat{K}_{\mathrm{II}}. \tag{G.38}$$

An accelerating observer in region I will naturally use $\hat{K}_{\mathrm{I}}$ as their Hamiltonian, while an accelerating observer in region II will naturally use $-\hat{K}_{\mathrm{II}}$ because the boost vector field points backwards in region II.

H Mode expansion cheat sheet

Here are the relationships between the three coordinate systems (t, x), (u, v), and (η, ξ):

$$t = \frac{1}{2}(v + u) = \xi \sinh(\eta) \tag{H.1}$$

$$x = \frac{1}{2}(v - u) = \xi \cosh(\eta) \tag{H.2}$$

$$u = t - x = -\xi e^{-\eta} \tag{H.3}$$

$$v = t + x = \xi e^{\eta} \tag{H.4}$$

$$\eta = \frac{1}{2} \ln\left(-\frac{t + x}{t - x}\right) = \frac{1}{2} \ln\left(-\frac{v}{u}\right) \tag{H.5}$$

$$\xi = \sqrt{x^2 - t^2} = \sqrt{-uv}. \tag{H.6}$$

The boost vector field is given by

$$K = x\partial_t + t\partial_x = -u\partial_u + v\partial_v = \partial_\eta. \tag{H.7}$$

The four sets of Rindler modes are

$$\begin{aligned} \Phi^{L,\lambda}_{\mathrm{I,IV}} &= (+v)^{-i|\lambda|}\Theta(+v) \\ \Phi^{L,\lambda}_{\mathrm{II,III}} &= (-v)^{+i|\lambda|}\Theta(-v) \\ \Phi^{R,\lambda}_{\mathrm{I,III}} &= (-u)^{+i|\lambda|}\Theta(-u) \\ \Phi^{R,\lambda}_{\mathrm{II,IV}} &= (+u)^{-i|\lambda|}\Theta(+u). \end{aligned} \tag{H.8}$$

Note that all of the above modes must have $\lambda > 0$ and are positive frequency with respect to Rindler time. Taking the complex conjugates of the above modes gives the negative frequency modes with respect to Rindler time. The superscripts L or R denote whether they are left or right moving. The subscripts denote which regions of the spacetime the modes are non-zero in.

The modes which are positive (and negative) frequency with respect to inertial time are

$$(v - i\epsilon)^{-i\lambda} = \Theta(v)v^{-i\lambda} + e^{-\pi\lambda}\Theta(-v)(-v)^{-i\lambda} = \Phi^{L,\lambda}_{\mathrm{I,IV}} + e^{-\pi\lambda}(\Phi^{L,\lambda}_{\mathrm{II,III}})^* \tag{H.9}$$

$$(v + i\epsilon)^{-i\lambda} = \Theta(v)v^{-i\lambda} + e^{+\pi\lambda}\Theta(-v)(-v)^{-i\lambda} = \Phi^{L,\lambda}_{\mathrm{I,IV}} + e^{+\pi\lambda}(\Phi^{L,\lambda}_{\mathrm{II,III}})^* \tag{H.10}$$

$$(u - i\epsilon)^{-i\lambda} = \Theta(u)u^{-i\lambda} + e^{-\pi\lambda}\Theta(-u)(-u)^{-i\lambda} = \Phi^{R,\lambda}_{\mathrm{II,IV}} + e^{-\pi\lambda}(\Phi^{R,\lambda}_{\mathrm{I,III}})^* \tag{H.11}$$

$$(u + i\epsilon)^{-i\lambda} = \Theta(u)u^{-i\lambda} + e^{+\pi\lambda}\Theta(-u)(-u)^{-i\lambda} = \Phi^{R,\lambda}_{\mathrm{II,IV}} + e^{+\pi\lambda}(\Phi^{R,\lambda}_{\mathrm{I,III}})^* \tag{H.12}$$

where the $-i\epsilon$ modes have positive inertial frequency while the $+i\epsilon$ modes have negative inertial frequency.

We write the Rindler modes as linear combinations of the above modes:

$$\Phi^{L,\lambda}_{\mathrm{I,IV}} = \frac{1}{e^{\pi\lambda} - e^{-\pi\lambda}}\left(e^{\pi\lambda}(v - i\epsilon)^{-i\lambda} - e^{-\pi\lambda}(v + i\epsilon)^{-i\lambda}\right) \tag{H.13}$$

$$\Phi^{R,\lambda}_{\mathrm{I,III}} = \frac{1}{e^{\pi\lambda} - e^{-\pi\lambda}}\left((u - i\epsilon)^{i\lambda} - (u + i\epsilon)^{i\lambda}\right) \tag{H.14}$$

$$\Phi^{L,\lambda}_{\mathrm{II,III}} = \frac{1}{e^{\pi\lambda} - e^{-\pi\lambda}}\left((v - i\epsilon)^{i\lambda} - (v + i\epsilon)^{i\lambda}\right) \tag{H.15}$$

$$\Phi^{R,\lambda}_{\mathrm{II,IV}} = \frac{1}{e^{\pi\lambda} - e^{-\pi\lambda}}\left(e^{\pi\lambda}(u - i\epsilon)^{-i\lambda} - e^{-\pi\lambda}(u + i\epsilon)^{-i\lambda}\right). \tag{H.16}$$

We now write the positive/negative frequency modes as linear combinations of plane waves:

$$(v - i\epsilon)^{-i\lambda} = \frac{e^{-\pi\lambda/2}}{\Gamma(i\lambda)} \int_0^\infty \frac{dk}{k} k^{i\lambda} e^{-ik(v-i\epsilon)} \tag{H.17}$$

$$(v + i\epsilon)^{-i\lambda} = \frac{e^{\pi\lambda/2}}{\Gamma(i\lambda)} \int_0^\infty \frac{dk}{k} k^{i\lambda} e^{ik(v+i\epsilon)} \tag{H.18}$$

$$(u - i\epsilon)^{-i\lambda} = \frac{e^{-\pi\lambda/2}}{\Gamma(i\lambda)} \int_0^\infty \frac{dk}{k} k^{i\lambda} e^{-ik(u-i\epsilon)} \tag{H.19}$$

$$(u + i\epsilon)^{-i\lambda} = \frac{e^{\pi\lambda/2}}{\Gamma(i\lambda)} \int_0^\infty \frac{dk}{k} k^{i\lambda} e^{ik(u+i\epsilon)}. \tag{H.20}$$

We now write the Rindler modes as linear combinations of plane waves.

$$\Phi^{L,\lambda}_{\mathrm{I,IV}} = \frac{1}{2\sinh(\pi\lambda)\Gamma(i\lambda)} \int_0^\infty \frac{dk}{k} k^{i\lambda} (e^{-ik(v-i\epsilon)} - e^{ik(v-i\epsilon)}) \tag{H.21}$$

$$\Phi^{R,\lambda}_{\mathrm{I,III}} = \frac{1}{2\sinh(\pi\lambda)\Gamma(-i\lambda)} \int_0^\infty \frac{dk}{k} k^{-i\lambda} (e^{\pi\lambda/2} e^{-ik(u-i\epsilon)} - e^{-\pi\lambda/2} e^{ik(u+i\epsilon)}) \tag{H.22}$$

$$\Phi^{L,\lambda}_{\mathrm{II,III}} = \frac{1}{2\sinh(\pi\lambda)\Gamma(-i\lambda)} \int_0^\infty \frac{dk}{k} k^{-i\lambda} (e^{\pi\lambda/2} e^{-ik(v-i\epsilon)} - e^{-\pi\lambda/2} e^{ik(v+i\epsilon)}) \tag{H.23}$$

$$\Phi^{R,\lambda}_{\mathrm{II,IV}} = \frac{1}{2\sinh(\pi\lambda)\Gamma(i\lambda)} \int_0^\infty \frac{dk}{k} k^{i\lambda} (e^{-ik(u-i\epsilon)} - e^{ik(u+i\epsilon)}). \tag{H.24}$$

We now write the plane waves in terms of our positive/negative frequency modes. Note that the below equations assume $k > 0$:

$$e^{-ik(v-i\epsilon)} = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \frac{\Gamma(i\lambda)}{e^{-\pi\lambda/2}} k^{-i\lambda} (v - i\epsilon)^{-i\lambda} \tag{H.25}$$

$$e^{ik(v+i\epsilon)} = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \frac{\Gamma(i\lambda)}{e^{\pi\lambda/2}} k^{-i\lambda} (v + i\epsilon)^{-i\lambda} \tag{H.26}$$

$$e^{-ik(u-i\epsilon)} = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \frac{\Gamma(i\lambda)}{e^{-\pi\lambda/2}} k^{-i\lambda} (u - i\epsilon)^{-i\lambda} \tag{H.27}$$

$$e^{ik(u+i\epsilon)} = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \frac{\Gamma(i\lambda)}{e^{\pi\lambda/2}} k^{-i\lambda} (u + i\epsilon)^{-i\lambda}. \tag{H.28}$$

In the above integrals, λ is being integrated through a pole at $\Gamma(0)$. The way to naivgate this pole is to add a tiny negative imaginary part to λ, so $\lambda \to \lambda - i\delta$ for $\delta > 0$. The above integrals can then be evaluated easily by deforming the λ contour and picking up the poles at $\lambda = in$ for $n = 0, 1, 2, \ldots$.

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